40th INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

Junior A-Level Paper, Fall 2018. Grades 8 – 9 (ages 13-15)

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed up.)

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points problems

1. In a triangle ABC, the point M is the midpoint of side BC. The point E is marked in the
5 segment AC so that $BE \geqslant 2AM$ (point E does not coincide with A or C). Prove that triangle
ABC is obtuse.

Nairi Sedrakyan

2. On an island with 2018 inhabitants each person is either a knight, or a liar, or a conformist.
Everyone knows about everyone who is who. One day all inhabitants of the island were arranged
in a line and every person answered in turn the same yes-or-no question "Are there more knights
than liars on the island?" Everybody heard all the previous answers. A knight always says the
6 truth, the liar always lies, and a conformist answers as the majority of people before him. In
case of the same number of "Yes" and "No" he chooses one of these answers at random. It
occurred that exactly 1009 inhabitants answered "Yes". Determine the greatest possible number
of conformists among the inhabitants of the island.

Mikhail Kuznetsov

3. One needs to write down a number of the form $77\dots7$ using only the digit 7. One can use
operations of addition, subtraction, multiplication, division, raising to the power, brackets, and
8 use any number of 7s in a row. Is there a number of the form $77\dots7$ which can be written down
in this way using a smaller number of 7s than in its decimal notation? For instance, the shortest
way to write down the number 77 is simply 77.

Sergey Markelov

4. A 7×7 grid board can be empty or can contain an invisible 2×2 ship, drawn along grid lines.
8 A detector placed in a cell of the board shows whether or not this cell is occupied by the ship.
All the detectors are switched on at the same moment. What is the smallest number of detectors
needed to determine if the ship is present on the board and if yes then where is it located?

Rustem Zhenodarov

5. An isosceles trapezoid $ABCD$ (AD parallel to BC) is inscribed into a circle with center O.
8 The line BO intersects the segment AD at the point E. If O_1 and O_2 are the centres of the circum-
circles of triangles ABE and BDE respectively, prove that the points O_1, O_2, O, C are concyclic.

Alexey Zaslavsky

6. Prove that

7 a) any integer of the form $3k-2$, where k is an integer, is the sum of a square and two cubes of
some integers;

3 b) any integer is the sum of a square and three cubes of some integers.

Nairi Sedrakyan

7. In a virtual world there are $n \geqslant 2$ towns. Some pairs of towns are connected by roads (no more than one road connects two towns). Any town can be reached from any other town by moving along the roads. One can change a road only in some town. The world is called *simple* if it is impossible starting from a town to return to this town without passing the same road twice. Otherwise the world is called *complicated*.

Peter and Basil play the following game. At the start, Peter chooses a single direction on every road, so that the road can be passed only in that direction, and places a tourist in one of the towns. Each turn Peter moves the tourist along a road in the permitted direction to a neighbouring town. On his turn, Basil changes the permitted direction on one of the roads adjacent to the town where the tourist is now. Basil wins if at some moment Peter cannot make a move. Prove that

5 a) in a simple world Peter can avoid loss, no matter how Basil plays;

7 b) in a complicated world Basil can guarantee his victory, no matter how Peter plays.

Maxim Didin

40th INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

Senior A-Level Paper, Fall 2018. Grades 10 – 11 (ages 15 and older)

(The result is computed from the three problems with the highest scores; the scores for the individual parts of a single problem are summed up.)

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points problems

5 1. On an island with 2018 inhabitants each person is either a knight, or a liar, or a conformist. Everyone knows about everyone who is who. One day all inhabitants of the island were arranged in a line and every person answered in turn the same yes-or-no question "Are there more knights than liars on the island?" Everybody heard all the previous answers. A knight always says the truth, the liar always lies, and a conformist answers as the majority of people before him. In case of the same number of "Yes" and "No" he chooses one of these answers at random. It occurred that exactly 1009 inhabitants answered "Yes". Determine the greatest possible number of conformists among the inhabitants of the island.

Mikhail Kuznetsov

7 2. In the acute non-isosceles triangle ABC, the point O is the center of the circumcircle and AH_a and BH_b are altitudes. The points X and Y are symmetric to the points H_a and H_b with respect to the midpoints of the sides BC and CA, respectively. Prove that the line CO divides the segment XY in half.

Folklore

3. Prove that

6 a) any integer of the form $3k-2$, where k is an integer, is the sum of a square and two cubes of some integers;

2 b) any integer is the sum of a square and three cubes of some integers.

Nairi Sedrakyan

8 4. A finite number of cells of an infinite grid are painted black, all other cells are white. Consider a paper polygon lying on the plane with sides along the grid lines containing at least two cells. This polygon may be translated (but not rotated) at any direction and distance, so that after a translation its sides are along the grid lines. If after a translation exactly one cell covered by the translated polygon is white, then this cell is painted black. Prove that there exists a white cell which will never be painted black, no matter of the number of translations.

Dmitry Zakharov

8 5. The three medians of a triangle divide its angles into six angles. What is the greatest possible number k of angles greater than $30°$ among these six angles?

Nairi Sedrakyan

9 6. On the real axis, an infinite number of positive integers are marked. When a wheel rolls along the axis, each marked point leaves a point trace on the wheel. Prove that one can choose R such that if a wheel of radius R starts from 0 and rolls along the line then every arc of size $1°$ will receive at least one point trace of a marked point.

Ivan Mitrofanov

7. Rockefeller and Marx play the following game. There are $n > 1$ cities, each with the same number of citizens. At the start of the game every citizen has exactly one coin (all coins are identical). On his turn, Rockefeller chooses one citizen from every city, then Marx redistributes their coins between them so that the new distribution is different from one immediately before. Rockefeller wins if at some moment there will be at least one citizen in every city with no coin. Prove that Rockefeller can always win, no matter how Marx plays, if in every city there are

10 a) $2n$ citizens;

4 b) $2n-1$ citizens.

Gleb Pogudin