39th INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Junior A-Level Paper, Fall 2017.

Grades 8 – 9 (ages 13-15)

(The result is computed from the three problems with the highest scores, the scores for the individual parts of a single problem are summed up.)

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points problems

4 **1.** You are given one metal weight weighing 6 kg, sugar and weightless bags for it in an unlimited amount. There is also a balance with two pans which are in equilibrium if the ratio of weights on the left and the right pan equals 3 : 4. For one weighing, you can place any already available weights on the balance, and then add a bag of sugar to one of the pans such that the balance will be in equilibrium. You can then use this bag of sugar for further weighings. Is it possible to get a bag with 1 kg of sugar?

Grigory Galperin

4 **2.** Given are two coins of radius 1 cm, two coins of radius 2 cm, and two coins of radius 3 cm. You may place two of the coins on a table so that they are tangent, and add other coins, one at a time, so that a newly placed coin is tangent to at least two of previously placed ones. A new coin cannot lie over an old one. Is it possible to place several coins on the table so that the centers of three of them are collinear for sure?

Egor Bakaev

6 **3.** An analyst made a prediction for the change in the dollar/euro rate for each of the next three months: by what percentage the rate would change in July, in August, and in September. It turned out that for every month, he predicted the right percentage but was mistaken if it will go up or down (i.e., if he predicted that the rate will decrease by x %, then the real rate increased by x %, and vice versa). Nevertheless, the dollar/euro rate after three months coincided with the prediction. Did the dollar/euro rate go up or down on the whole?

Alexey Zaslavsky

4. There are 100 doors, each with its own key (which opens only this door). The doors are numbered 1 through 100, and so are the keys. It is known that the number of every key is either equal to the number of the door which it opens, or differs by 1. For one attempt you can select any door and any key and check whether a chosen key opens a chosen door. Is it always possible to find out which key opens which door:

1 a) in 99 attempts;

3 b) in 75 attempts;

4 c) in 74 attempts?

Alexey Lebedev, Alexandr Shapovalov

9 **5.** The decimal digits of a positive integer $n > 1$ are written in reverse order, and the resulting number is multiplied by n. Is it possible that we get a number with all digits equal to 1?

Fedor Petrov

9 **6.** The incircle of the triangle ABC touches the sides AB, BC and AC of the triangle ABC at points N, K and M, respectively. The lines MN and MK intersect the external bisector of angle ABC at points R and S, respectively. Prove that the lines RK and SN intersect on the incircle of the triangle ABC.

Mikhail Evdokimov

7. A city is a rectangle divided into equal square blocks with a single building in each block. Each building has 5 floors. The law of renovation allows you to choose two blocks with a common side, which contain buildings at this moment, and demolish that building which has fewer floors (or the same number of floors). After the demolition, the number of floors that were in the demolished building is added to the remaining building. What is the smallest possible number of buildings that can be left in the city by using the law of renovation, if the city consists of

5 a) 20×20 blocks;

5 b) 50×90 blocks?

Mikhail Murashkin

39th INTERNATIONAL MATHEMATICAL TOURNAMENT OF TOWNS

Senior A-Level Paper, Fall 2017.

Grades 10 – 11 (ages 15 and older)

(The result is computed from the three problems with the highest scores, the scores for the individual parts of a single problem are summed up.)

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points problems

1. There are 100 doors, each with its own key (which opens only this door). The doors are numbered 1 through 100, and so are the keys. It is known that the number of every key is either equal to the number of the door which it opens, or differs by 1. For one attempt you can select any door and any key and check whether a chosen key opens a chosen door. Is it always possible to find out which key opens which door:

1 a) in 99 attempts;

2 b) in 75 attempts;

3 c) in 74 attempts?

Alexey Levedev, Alexandr Shapovalov

2. Six circles of radius 1 with centers in the vertices of a regular hexagon are drawn, so that
5 the center O of the hexagon lies inside all six circles. An angle with angular measure α
and vertex O cuts out six arcs in these circles. Prove that the sum of the sizes of these arcs is equal to 6α.

Egor Bakaev

3. An analyst made a prediction for the change in the dollar/euro rate for each of the next 12 months: by what percentage the rate would change in October, in November, in December, and so on. It turned out that for every month, he predicted the right
6 percentage but was mistaken if it will go up or down (i.e., if he predicted that the rate
will decrease by x %, then the real rate increased by x %, and vice versa). Nevertheless, the dollar/euro rate after 12 months coincided with the prediction. Did the dollar/euro rate go up or down on the whole?

Alexey Zaslavsky

4. Show that for any infinite sequence $a_0, a_1, \ldots, a_n, \ldots$ of ones and negative ones, we can choose n and k such that

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$$|a_0 \cdot a_1 \cdot \ldots \cdot a_k + a_1 \cdot a_2 \cdot \ldots \cdot a_{k+1} + \ldots + a_n \cdot a_{n+1} \cdot \ldots \cdot a_{n+k}| = 2017.$$

Ivam Mitrofanov

5. You must cut a piece of cheese into parts following the rules: 1) The first cut must divide the cheese into two pieces, every next cut divides one of the existing pieces into two; 2) after every cut, the ratio of the weight of any piece to the weight to any other one must be greater than a given number R.

3 a) Prove that for $R = 0.5$ we can cut the cheese so that the process will never stop (i.e., after any number of cuts, we will still be able to make one more cut).

4 b) Prove that if $R > 0.5$, then at some point we will have to stop cutting.

4 c) What is the greatest number of parts we can achieve if $R = 0.6$?

Alexey Tolpygo

6. A triangle ABC is given. Let I be the center of its excircle tangent to the segment AB,
10 and let A_1 and B_1 be the points where the segments BC and AC touch the corresponding
excircles. Let M be the midpoint of the segment IC, and let the segments AA_1 and BB_1 intersect at point N. Prove that the points N, B_1, A, and M are concyclic.

Fedor Ivlev

7. A city has the form of an $n \times n$ square divided into 1×1 blocks. The streets run from north to south and from west to east. A man walks every day from the south-west corner
10 to the north-east corner, moving only north or east, and then returns, moving only south
or west. Each time (going forth and going back) he chooses his path so that the total length of its parts previously passed in any direction is minimal. Prove that in n days he will pass all the length of all streets.

Maxim Didin