37th INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

Junior A-Level Paper, Spring 2016

Grades 8 – 9 (ages 13-15)

(The result is computed from the three problems with the highest scores, the scores for the individual parts of a single problem are summed up.)

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points	problems	
4	1.	All integers from 1 to one million are written on a tape in some arbitrary order. Then the tape is cut into pieces containing two consecutive digits each. Prove that these pieces contain all two-digit integers for sure, regardless of the initial order of integers. *Alexey Tolpygo*
	2.	Do there exist integers a and b such that
2	a)	the equation $x^2 + ax + b = 0$ has no real roots, and the equation $[x^2] + ax + b = 0$ has at least one real root?
3	b)	the equation $x^2 + 2ax + b = 0$ has no real roots, and the equation $[x^2] + 2ax + b = 0$ has at least one real root? (By $[k]$ we denote the integer part of k, that is, the greatest integer not exceeding k.) *Alexandr Khrabrov*
6	3.	Given a square with side 10. Cut it into 100 congruent quadrilaterals such that each of them is inscribed into a circle with diameter $\sqrt{3}$. *Ilya Bogdanov*
8	4.	A designer took a wooden cube $5 \times 5 \times 5$, divided each face into unit squares and painted each square black, white or red so that any two squares with a common side have different colours. What is the least possible number of black squares? (Squares with a common side may belong to the same face of the cube or to two different faces.) *Mikhail Evdokimov*
8	5.	Let p be a prime integer greater than 10^k. Pete took some multiple of p and inserted a k-digit integer A between two of its neighbouring digits. The resulting integer C was again a multiple of p. Pete inserted a k-digit integer B between two of neighbouring digits of C belonging to the inserted integer A, and the result was again a multiple of p. Prove that the integer B can be obtained from the integer A by a permutation of its digits. *Ilya Bogdanov*
9	6.	An automatic cleaner of the disc shape has passed along a plain floor. For each point of its circular boundary there exists a straight line that has contained this point all the time. Is it necessarily true that the center of the disc stayed on some straight line all the time? *Izyaslav Vainshtein*
	7.	
5	a)	There are $2n + 1$ ($n > 2$) batteries. We don't know which batteries are good and which are bad but we know that the number of good batteries is greater by 1 than the number of bad ones. A lamp uses two batteries, and it works only if both batteries are good. What is the least number of attempts sufficient to make the lamp work for sure?
5	b)	The same problem but the total number of batteries is $2n$ ($n > 2$) and the numbers of good and bad batteries are equal. *Alexandr Shapovalov*

37th INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

Senior A-Level Paper, Spring 2016

Grades 10 – 11 (ages 15 and older)

(The result is computed from the three problems with the highest scores, the scores for the individual parts of a single problem are summed up.)

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points	problems	
4	1.	All integers from 1 to one million are written on a tape in some arbitrary order. Then the tape is cut into pieces containing two consecutive digits each. Prove that these pieces contain all two-digit integers for sure, regardless of the initial order of integers. *Alexey Tolpygo*
5	2.	Given a square with side 10. Cut it into 100 congruent quadrilaterals such that each of them is inscribed into a circle with diameter $\sqrt{3}$. *Ilya Bogdanov*
6	3.	Let M be the midpoint of the base AC of an isosceles triangle ABC. Points E and F on the sides AB and BC respectively are chosen so that $AE \neq CF$ and $\angle FMC = \angle MEF = \alpha$. Determine $\angle AEM$. *Maxim Prasolov*
8	4.	There are 64 towns in a country, and some pairs of towns are connected by roads but we don't know these pairs. We may choose any pair of towns and find out whether they are connected by a road. Our aim is to determine whether it is possible to travel between any two towns using roads. Prove that there is no algorithm which would enable us to do this in less than 2016 questions. *Konstantin Knop*
8	5.	On a blackboard, several polynomials of degree 37 are written, each of them has the leading coefficient equal to 1. Initially all coefficients of each polynomial are non-negative. By one move it is allowed to erase any pair of polynomials f, g and replace it by another pair of polynomials f_1, g_1 of degree 37 with the leading coefficients equal to 1 such that either $f_1+g_1 = f+g$ or $f_1g_1 = fg$. Prove that it is impossible that after some move each polynomial on the blackboard has 37 distinct positive roots. *Alexandr Kuznetsov*
	6.	Recall that a palindrome is a word which is the same when we read it forward or backward.
4	a)	We have an infinite number of cards with words "*abc*", "*bca*", "*cab*". A word is made from them in the following way. The initial word is an arbitrary card. At each step we obtain a new word either gluing a card (from the right or from the left) to the existing word or making a cut between any two of its letters and gluing a card between both parts. Is it possible to obtain a palindrome this way?
6	b)	We have an infinite number of red cards with words "*abc*", "*bca*", "*cab*" and of blue cards with words "*cba*", "*acb*", "*bac*". A palindrome was formed from them in the same way as in part (a). Is it necessarily true that the number of red and blue cards used was equal? *Alexandr Gribalko, Ivan Mitrofanov*
	7.	A spherical planet has the equator of length 1. On this planet, N circular roads of length 1 each are to be built and used for several trains each. The trains must have the same constant positive speed and never stop or collide. What is the greatest possible sum of lengths of all the trains? The trains are arcs of zero width with endpoints removed (so that if only endpoints of two arcs have coincided then it is not a collision). Solve the problem for
4	a)	$N = 3$;
6	b)	$N = 4$. *Alexandr Berdnikov*