37th INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

Junior A-Level Paper, Fall 2015.

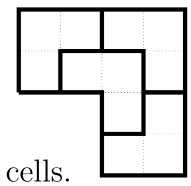
Grades 8 – 9 (ages 13-15)

(The result is computed from the three problems with the highest scores, the scores for the individual parts of a single problem are summed up.)

---

points problems

1. A grid polygon is called *amazing* if it is not a rectangle and several its copies can form a polygon similar to it. For instance, a corner consisting of three cells is an amazing polygon (see the figure on the right).



2 a) Find an amazing polygon consisting of 4 cells.

3 b) Determine all $n > 4$ such that there exists an amazing polygon consisting of n cells.

Egor Bakaev

2. A set consists of all integers from 1 to 100 except some k integers. Is it always possible to choose k distinct integers in this set so that their sum equals 100 if

2 a) $k = 9$;

4 b) $k = 8$?

Alexandr Shapovalov

3. Prove that the sum of lengths of any two medians in an arbitrary triangle is

3 a) not greater than $3P/4$ where P is the perimeter of this triangle;

5 b) not less than $3p/4$ where p is the semiperimeter of this triangle.

Lev Emelyanov

8 4. A 9×9 grid square is made of matches, every cell side consists of a single match, and any two adjacent cells share exactly one match. Pete and Basil in turn remove matches one by one. A player wins if there remains no entire 1×1 square after his move. Who of the players has a winning strategy?

Alexandr Shapovalov

8 5. In triangle ABC, medians AA_0, BB_0 and CC_0 intersect at point M. Let P, Q, R, and T be the circumcenters of triangles MA_0B_0, MCB_0, MA_0C_0, MBC_0 respectively. Prove that points P, Q, R, T, M are concyclic.

Pavel Kozhevnikov

6. Several distinct real numbers are written on a blackboard. Peter wants to make an expression such that its values are exactly these numbers. To make such an expression, he may use any real numbers, brackets, and usual signs $+$, $-$ and $\times$. He may also use a special sign $\pm$: computing the values of the resulting expression, he chooses values $+$ or $-$ for every $\pm$ in all possible combinations. For instance, the expression 5 ± 1 results in $\{4, 6\}$, and $(2 \pm 0.5) \pm 0.5$ results in $\{1, 2, 3\}$. Can Pete construct such an expression:

3 a) if the numbers on the blackboard are 1, 2, 4;

7 b) for any collection of 100 distinct real numbers on a blackboard?

Koh, Bong-Gyun (South Korea)

10 7. Santa Claus had n sorts of candies, k candies of each sort. He distributed them at random between k gift bags, n candies in each, and gave a bag to each of k children. The children learned what they had in the bags and decided to trade. Two children can trade one candy for one candy in case if each of them gets a candy of the sort that he/she lacks. Is it true that a sequence of trades can always be arranged so that in the end every child has candies of each sort?

Mikhail Evdokimov

37th INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

Senior A-Level Paper, Fall 2015.

Grades 10 – 11 (ages 15 and older)

(The result is computed from the three problems with the highest scores, the scores for the individual parts of a single problem are summed up.)

---

points problems

3 1. A geometrical progression consists of 37 positive integers. The first and the last terms are relatively prime numbers. Prove that the 19th term of the progression is the 18th power of a positive integer.

Boris Frenkin

6 2. A 10×10 grid square is split by 80 unit grid segments (lying inside the square) into 20 polygons of equal area. Prove that all these polygons are congruent.

Pavel Kozhevnikov

6 3. Each coefficient of a non-constant polynomial is an integer of absolute value not exceeding 2015. Prove that every positive root of this polynomial is greater than $1/2016$.

Alexandr Khrabrov

7 4. Suppose that a quadrilateral $ABCD$ is cyclic. Let extensions of the opposite sides intersect at points P and Q, and let K and N be the midpoints of the diagonals. Prove that $\angle PKQ + \angle PNQ = 180°$.

Maxim Didin

5. Several distinct real numbers are written on a blackboard. Peter wants to make an expression such that its values are exactly these numbers. To make such an expression, he may use any real numbers, brackets, and usual signs $+$, $-$ and $\times$. He may also use a special sign $\pm$: computing the values of the resulting expression, he chooses values $+$ or $-$ for every $\pm$ in all possible combinations. For instance, the expression 5 ± 1 results in $\{4, 6\}$, and $(2 \pm 0.5) \pm 0.5$ results in $\{1, 2, 3\}$. Can Pete construct such an expression:

2 a) if the numbers on the blackboard are 1, 2, 4;

6 b) for any collection of 100 distinct real numbers on a blackboard?

Koh, Bong-Gyun (South Korea)

6. Basil has a watermelon in a shape of a ball with diameter 20 cm. Using a long knife, Basil makes three pairwise perpendicular cuts, each cut is of depth h (a cut produces a circular segment with height h in the plane of the cut) . Does it necessarily follow that the watermelon is divided into two or more pieces if

6 a) $h = 17$ cm;

6 b) $h = 18$ cm?

Mikhail Evdokimov

12 7. N children, no two of the same height, stand in a line in some order. The following two-step procedure is applied repeatedly: firstly, the line is split into the least possible number of groups so that in each group all children are arranged from the left to the right in ascending order of the height (a group may consist of a single child). Secondly, the order of children in each group is changed to the opposite one (so now in each group the children stand in descending order). Prove that after $N - 1$ rearrangements the children in the line will stand in descending order from the left to the right.

Nikita Gladkov