# Dynamics of Tilings

Problem 10ef

Problem **4c** for any non-rational plane. Let $A$ be a figure in the space. We say that the *fractional part* $\{A\}$ of the figure $A$ is the set of all points of the form $(\{x\},\{y\},\{z\})$, where $(x,y,z)$ lies in $A$.

The *winding* of the cube $[0,1)^3$ with a plane $L$ is the fractional part of $L$.

A *tile* of the winding is the fractional part of any tile in the plane $L$. As in the lower-dimensional case, the tiles of the winding are the sections of the cube by the planes parallel to $L$. Moreover, in the case of an irrational plane the tiles of the winding are in one-to-one correspondence with the tiles on the plane.

Proposition. The winding with any irrational or semirational plane meets any segment in the cube $[0,1)^3$ which is not parallel to the plane. Moreover, there are infinitely many such meeting points.

*Proof.* Due to a permutation of the coordinates, we may assume that the plane is not parallel to the $z$-axis, so it is determined by an equation of the form $z=\mu x+\lambda y$. At least one of $\mu$ and $\lambda$ (say, $\lambda$) is irrational, otherwise the plane would be rational.

Suppose first that the given segment $I$ is vertical, so it is the subset of a vertical line $V$ (given by $x=x_0$, $y=y_0$) determined by the conditions $p<z<q$. The points of the plane whose fractional parts lie on $V$ have the form $(x,y,\mu x+\lambda y)=\big(x_0+k,y_0+m,\mu(x_0+k)+\lambda(y_0+m)\big)$ for some integer $k$ and $m$. Therefore, the points of $V$ lying in the winding are exactly those having $z$-coordinates of the form $z=\{(\mu x_0+\lambda y_0)+k\mu+m\lambda\}$. Fix an arbitrary $k$; then the Dirichlet lemma guarantees that this number lies in $(p,q)$ for some integer $m$.

In the general case, when $I$ has an arbitrary direction, it still contains some subsegment which can be projected along the plane onto some vertical segment $J$ in the cube. Since $J$ meets the winding, this subsegment also does.

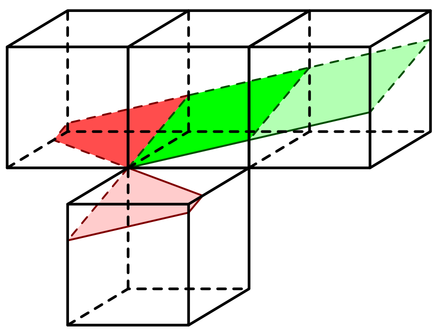
Finally, since each segment $I$ contains infinitely many non-overlapping subsegments each of which meets the winding, there are infinitely many meeting points of $I$ with the winding. □

Problem **3** for non-rational planes. Firstly, we will formulate and prove all subproblems for an irrational plane; then we will make two easy observations which allow to solve all these problems for the semirational case.

Problem **3a** for a non-rational plane: Can there be two congruent tiles?

Answer: Yes.

E.g., consider an irrational (or semirational) plane $z=\mu x+\lambda y$ with positive coefficients $\mu<\frac{1}{4}$ and $\lambda<\frac{1}{3}$. The intersections of this plane with the cube $[0,1)^3$ and with its shift by 1 to the right are two congruent parallelograms. Also, together with every tile, this plane contains its congruent copy obtained by the reflection with respect to the origin.



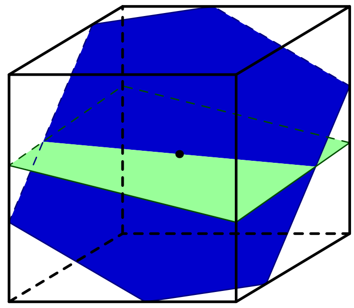
Problem **3b** for an irrational plane: can there be or not be infinitely many pairwise congruent tiles?

Answer is different from the one-dimensional case: both options are now possible. An irrational plane may happen to contain infinitely many congruent tiles, and may fail to contain such.

To construct both examples, we notice that the directions of a planes intersecting the cube can be of one of the two types described in the following lemma.

Lemma 1. Assume that a plane $L$ contains the center of the cube and does not contain any of its vertices; the plane $L$ partitions the set of vertices of the cube into two 4-tuples. Each of these 4-tuples forms **either** the set of vertices of some facet (in this case we say that $L$ is a *4-plane*, and its direction is a *4-direction*), **or** the set of a vertex and three its neighbors (then we say that $L$ is a *6-plane*, and its direction is a *6-direction*).

We assigned such names because the section of the cube by $L$ is a centrally symmetrical quadrilateral in the former case, and a centrally symmetrical quadrilateral in the latter one.



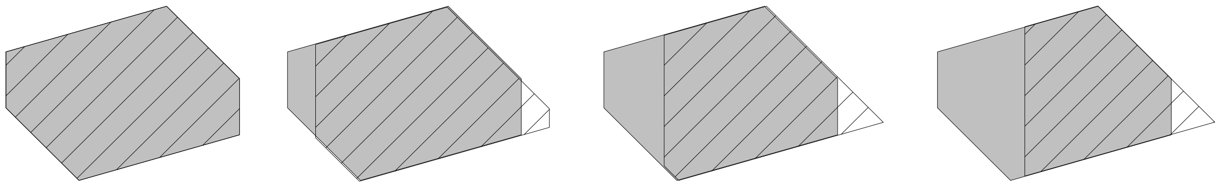
*Proof.* Any 4-tuple of vertices of the cube lying on one side of $L$ cannot contain two opposite vertices of the cube. There are only three types of the 4-tuples satisfying this condition: the two mentioned in the Lemma formulation, along with a "chess-like" tuple containing no adjacent vertices. But the "chess-like" 4-tuple also cannot lie on one side of $L$, since the center of the cube lies inside the tetrahedron $ABCD$. □

Notice that both types actually exist. An example of a 4-plane is presented above in **3a**, and the perpendicular bisector of the diagonal of the cube is an example of a 6-plane. In general, a plane $Ax+By+Cz=0$ has a 6-direction (with respect to the cube $[0,1]^3$) if the maximum of $|A|$, $|B|$, and $|C|$ does not exceed the sum of the two other numbers, and it has a 4-direction otherwise.

Now, to solve the planar analogue of **3b**, consider first a non-rational 4-plane $L$ containing the center of the cube $[0,1]^3$. By **4c**, the winding with $L$ meets at infinitely many points each small segment perpendicular to $L$ and passing through the center. If this segment is short enough, each meeting point belongs to a tile which is close enough to the section of the cube by $L$, and each such tile is a parallelogram congruent to this central section.
Conversely, each irrational 6-plane $L$ cannot contain more than two congruent tiles. This follows from the following fact.

Lemma 2. Any two nonempty sections of the cube $[0,1]^3$ by planes having a given 6-direction have different areas (and thus are incongruent), unless they are symmetric to each other with respect to the center of the cube.

*Proof.* In fact, we will prove a bit more. Let us draw a 6-plane $L$, and then shift it continuously increasing the distance to the center. Then the area of the section of the cube will strictly decrease until it vanishes.
To prove this monotonicity, we switch to a different reference system. Instead of shifting the plane, let us shift the cube along the $z$-axis at distance $h$ and check what happens with its section $S_h$ by the fixed plane $L$. As $h$ increases, we observe the following phases:



In each figure above, the solid polygon is the initial section $S_0$, while the hatched one shows a section $S_h$ (the values of $h$ increase from the left to the right). Any hexagonal section $S_h$ is obtained from $S_0$ by removing of a trapezoid and pasting another one. The altitudes of the trapezoids are equal, and the small base of the removed trapezoid is congruent to the large base of the pasted one. Thus the area of $S_h$ is smaller than that of $S_0$. Moreover, as $h$ increases, the altitudes of the trapezoids also increase, thus so does the difference of their areas. Therefore, $S_h$ decreases for such values of $h$.
Next, after the section abandons being a hexagon, for any $h > g$ the section $S_h$ merely lies in $S_g$, so it has smaller area. □

Remark. The arguments from the proofs of the two lemmas above are quite helpful to complete the solution of **10c**.

Problem **3c** for an irrational plane: can there be or not be three congruent tiles incongruent to all the other ones?

Answer: There cannot be such tiles on an irrational plane.

Indeed, by Lemma 2, the winding with any 6-plane contains at most two congruent copies of any tile (including the tile itself).
Similarly, one may prove the analogue of Lemma 2 for a 4-plane: Any two nonempty sections of the cube $[0,1]^3$ by planes having a given 4-direction have different areas (and thus are incongruent), unless they either are symmetric to each other with respect to the center of the cube, or are both congruent to the central section of the cube by the plane of our direction.
This statement shows that the winding with any 4-plane contains infinitely many tiles congruent to the central one, and at most two congruent copies of any other tile.

Problem **3d** for an irrational plane: Can there be or not be infinitely many pairwise incongruent tiles?

Answer: There always are infinitely many pairwise incongruent tiles on any irrational plane.

Indeed, according to **4c**, the winding with our plane meets infinitely many times any sufficiently small segment perpendicular to the plane and starting at a vertex of the cube towards the cube. Each such meeting point belongs to a triangular tile, all these tiles are similar to each other with ratios different from 1.

Problems **3a-d** for semirational plane: By definition, a semirational plane maps to itself under some shift $T$ by a nonzero integer vector. Thus each tile on such plane is congruent to infinitely many tiles obtained from it by iterative application of $T$.
Due to this property, a semirational plane is close to rational ones in some aspects. On the other hand, it differs from rational planes because the problem **4c** is still applicable. Thus the solutions based on an application of **4c** work for semirational planes as well.
These two observations yield the following answers in the semirational case: (**a**) they can (and should); (**b**) there always are such tiles; (**c**) there cannot be such tiles; (**d**) there always are such tiles.

Problem **3e** for a non-rational plane: Describe in geometrical terms all the planes (not necessarily containing 0) which contain two congruent tiles which are incongruent to all other tiles.

Answer: These are all irrational planes containing a point with half-integer coordinates.

Indeed, semirational and rational planes do not fit since they contain infinitely many congruent copies of each tile. On the other hand, if an irrational plane contains exactly two congruent copies of some tile, then these copies are centrally symmetric to each other (recall Lemma 2 for 6-planes and its duplicate for 4-planes which has been mentioned in the analogue of **3c**). Now, as in a similar one-dimensional problem, one can easily see that the center of symmetry has half-integer coordinates.

Problem 11a

We will state and prove the following *two-dimensional Weyl's lemma.* Assume that at least one of the numbers $\lambda$ and $\mu$ is irrational. Denote by $W_N$ the tuple of the numbers $\{b+k\mu+n\lambda\}$ for all positive integer $k$ and $n$ that do not exceed $N$. This tuple consists of $N^2$ elements (some of them may occur more than once). Then, the frequency of hits of the tuple $W_N$ in the segment $I$ tends to the length of the segment $|I|$ as $N \to \infty$.

Without loss of generality, we may assume that $\lambda$ is irrational. Note that in the solution of Problem **7d** we proved a slightly stronger result: for any irrational $\lambda$ and positive $\varepsilon$ there exists $N_0$ satisfying the following condition. For any $a$, if we take $N > N_0$ initial terms of the Weyl's sequence $\{a+n\lambda\}$, then the difference between their frequency of hits in $I$ and the length $|I|$ will be less than $\varepsilon$. The strength of this statement consists in the fact that the same number $N_0$ is suitable for any $a$.

For every $N > N_0$, the tuple $W_N$ is split into the initial length $N$ segments of the Weyl's progressions with the difference $\lambda$ starting from terms of the form $b+k\mu$. For each of those initial segments, the frequency of hits in $I$ differs from $|I|$ by less than $\varepsilon$. Therefore, Corollary 1 implies that the same holds for the frequency of hits in $I$ of the entire tuple $W_N$.

Remark. In fact, one can prove a more general fact. On a plane with coordinates $(k,n)$, choose an arbitrary convex polygon $P$. By $P_N$ denote its image under the homothety with ratio $N$. Then, we can define "the initial segment" $W_N$ of the two-parameter sequence $\{b+k\mu+n\lambda\}$ to be the set of the terms with $(k,n)$ contained in $P_N$. For instance, if we set $P=[0,1]^2$, then the "initial segment" $W_N$ will be exactly the same as the one defined in the solution above. It turns out that no matter which convex polygon $P$ we take, the frequency of hits of the tuple $W_N$ in the segment $I$ tends to $|I|$ as $N \to \infty$. We encourage the reader to try to show it.

Problem 11b

Since the plane $Ax+By+Cz=0$ is irrational, it does not contain the vertical axis, therefore, we have $C \neq 0$. Then, the set of all the points $(x,y,z)$ on this plane such that $x$ and $y$ belong to the interval $[0,N]$ is a parallelogram, which we will denote $P_N$.

We will formulate and prove the following analog of Problem 8. Choose a segment $I$ in the cube $[0,1]^3$. By $a,b,c$ denote the differences of the coordinates of its endpoints. Then, we have the following equalities:

$$\lim_{N\to\infty} \frac{\text{the number of intersection points of } \{P_N\} \text{ with the segment } I}{\text{the area of } P_N} = \frac{|Aa+Bb+Cc|}{\sqrt{A^2+B^2+C^2}} =$$

$$= \text{the length of the orthogonal projection of the segment } I \text{ onto the line orthogonal to the plane.} \qquad (*)$$

The second equality here is obvious. Indeed, the vector $v$ with coordinates $(A,B,C)$ is orthogonal to the plane. Denote the angle between $v$ and the vector $I$ by $\alpha$. We obtain that the middle part of $(*)$ equals $\dfrac{|v\cdot I|}{|v|} = \dfrac{|v||I|\cos\alpha}{|v|} = |I|\cos\alpha$, that is, the length of orthogonal projection of $I$ onto a line parallel to $v$.

To prove the first equality in $(*)$, we will need the following observations.

Firstly, note that the statement holds for vertical segment. Indeed, since $C \neq 0$, we can rewrite the equation of the plane as $z = \lambda x + \mu y$, and at least one of the coefficients $\lambda$ and $\mu$ is irrational (otherwise, the plane is rational). Then, for the segment $I$ defined by $x = x_0$, $y = y_0$ and $p < z < q$, the statement is exactly the two-dimensional Weyl's theorem for the progression $\{(\lambda x_0 + \mu y_0) + \lambda k + \mu n\}$ and the segment $[p, q]$.

We should not forget that in Weyl's theorem, the denominator of the fraction under the limit is the number of pairs $(k, n)$ in the square $[1, N]^2$, while the one in the sought limit equals the area of the parallelogram $P_N$. Hence, we should divide the answer given by Weyl's theorem by the quotient of the number of pairs $N^2$ and the area of $P_N$. Obviously, the latter does not depend on $N$, thus, it equals the area of $P_1$, that is, $\frac{1}{\sqrt{1+\lambda^2+\mu^2}}$. Thus, we have proved $(*)$ for $I$ being vertical.

Secondly, note that if segments $QR$ and $ST$ in the cube $[0, 1]^3$ are parallel to the plane $Ax + By + Cz = 0$, then the equality $(*)$ for the segment $QS$ is equivalent to $(*)$ for $RT$. Indeed, the left-hand sides of those equalities coincide, since any plane parallel to $Ax + By + Cz = 0$ meets or does not meet both $QS$ and $RT$ simultaneously. The right-hand sides of the equalities coincide, since they are both equal to the lengths of the orthogonal projections of $QS$ and $RT$ on the line orthogonal to $Ax + By + Cz = 0$, and those projections coincide.

Thirdly, note that if an arbitrary segment $I$ is divided into several parts, then the equality $(*)$ for $I$ follows from the equalities $(*)$ for its parts (namely, it will be the sum of those equalities). We will call two segments $QS$ and $RT$ *analogous*, if the segments $QR$ and $ST$ are parallel to the plane $Ax + By + Cz = 0$. Note that an arbitrary segment $I$ in $[0, 1]^3$ can be split into parts analogous to vertical segments (for instance, the planes parallel to the given one which are drawn through each the vertices of the cube divide $I$ in a proper way). According to the remarks above, the equality $(*)$ holds for vertical segments. Therefore, it holds for the parts of $I$ which are analogous to them, and thus, it holds for the entire segment $I$.

Remark. Had we chosen at the beginning another coordinate to be vertical, the parallelogram $P_N$ employed in $(*)$ would be a different one. However, the left-hand side of the equality $(*)$ would not change, since it equals the right-hand side, which does not depend on the choice of $P_N$. In fact, the left-hand side of $(*)$ will not change even if we replace the family of parallelograms $P_N$ by the homothetic images of an arbitrary convex polygon. We encourage the reader to try to prove it using the remark in the solution of the previous problem.

## Problem 11c

We define the density of tiles on an irrational plane $Ax + By + Cz = 0$ to be the limit of the fraction

$$\frac{\text{number of tiles contained in } P_N}{\text{the area of } P_N},$$

where $P_N$ is a parallelogram defined in the solution of the previous problem.

For the sake of convenience, assume first that $A, B$ and $C$ are positive. Then, the number of tiles contained in $P_N$ equals the number of the intersection points of its fractional part $\{P_N\}$ with the cube diagonal connecting the vertices $(0, 0, 0)$ and $(1, 1, 1)$. Applying the result of the previous problem to this diagonal, we obtain that the sought density equals

$$\frac{A + B + C}{\sqrt{A^2 + B^2 + C^2}}.$$

To obtain the general answer, one should replace the numerator by $|A| + |B| + |C|$.

## Problem 11d

In analogy with the derivation of Problem **9b** from **9a**, we obtain that the answer is the multiplicative inverse of the answer to the previous problem.

## Problem 11e

We define the density of triangular tiles on an irrational plane $Ax+By+Cz=0$ to be the limit of the fraction

$$\frac{\text{the number of triangular tiles inside } P_N}{\text{the area of } P_N}.$$

Assume that $A, B$ and $C$ are positive. For every vertex of the cube, draw a plane passing through it and parallel to $Ax+By+Cz=0$. Those planes divide the diagonal connecting the vertices $(0,0,0)$ and $(1,1,1)$ into segments, by $I$ and $J$ we denote the marginal ones. They are of the same length, and the ratio of this length to the length of the diagonal equals $\min(A,B,C)/(A+B+C)$. Note that the plane $Ax+By+Cz=D$ intersects the cube $[0,1]^3$ by a triangle if and only if it meets one of the segments $I$ or $J$, therefore, the ratio of the sought density of triangular tiles to the known density of all the tiles is $2\min(A,B,C)/(A+B+C)$.
So, in the general case the answer is

$$\frac{2\min(|A|,|B|,|C|)}{\sqrt{A^2+B^2+C^2}}.$$

Problem 11f

The sought probability equals the quotient of the densities of the triangular tiles and all the tiles, that is,

$$\frac{2\min(|A|,|B|,|C|)}{|A|+|B|+|C|}.$$

Problem 11g

Through every vertex of the cube $[0,1]^3$, draw a plane parallel to the given one. These planes divide the cube into several parts, and two of the parts are tetrahedra congruent to each other.
It follows from the solution of Problem **17** that the answer equals to the sum of the volumes of those two tetrahedra, that is, $\dfrac{|ABC|}{3\max(|A|,|B|,|C|)^3}$.

Polyhedra

Let $P$ and $Q$ be polyhedra expressed as a union of (closed) convex polygons $C_1, C_2, \ldots$ and $D_1, D_2, \ldots$, respectively.

Problem 12a

The union of $P$ and $Q$ can be expressed as the union of the polygons $C_1, C_2, \ldots, D_1, D_2, \ldots$, and the intersection of $P$ and $Q$ can be expressed as the union of all pairwise intersections $C_i \cap D_j$, that are (closed) convex polytopes.
With each of its points $x$, a component of $P$ contains all the polygons $C_i$ such that $x \in C_i$. Therefore, each of the components of $P$ is the union of some of the polygons $C_1, C_2, \ldots$.

Problem 12b

The complement to each of the convex polygons $C_i$ is a polyhedron, that we will denote by $\overline{C_i}$. Indeed, if the polygon is defined an the intersection of several half-planes, then the complement to it is the union of the complements to those half-planes. Thus, the complement to a polyhedron $P$ is the intersection of the $\overline{C_i}$, which is itself a polyhedron, as it follows from the previous problem.

Problem 12c

Let $P$ and $Q$ be polyhedra having $p$ and $q$ components, respectively. Then, their union can have any number of components from 1 to $p+q$, but not more. Indeed, the number of components does not exceed $p+q$, because each component of the union of $P$ and $Q$ contains a component of $P$ or $Q$.

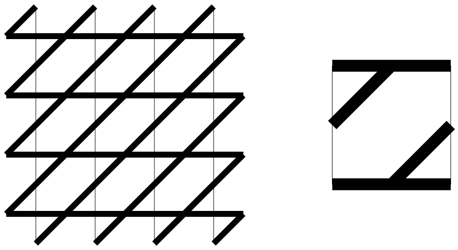
We shall construct the examples for the maximal and the minimal possible number of components. For the rest of cases, the examples can be constructed in a similar way. Define $P$ to be the set $[0;1]\cup\{2,3,\dots,p\}$. The set $Q_1$ consists of $q$ points $\{p+1,p+2,\dots,p+q\}$ and its union with $P$ has $p+q$ components, while the union of $Q_2=[1;p]\cup\{\frac{1}{2},\frac{1}{3},\dots,\frac{1}{q}\}$ with $P$ is connected.
At the same time, one can easily construct examples showing that the intersection of $P$ and $Q$ and the complement to $P$ may have any number of components for any $p$ and $q$. For instance, take $P$ being a union of one horizontal and $N$ vertical lines, and $Q$ being an inclined line, then, the intersection $P\cap Q$ and the complement to $P$ have a lot of components (more than $N$ and $2N$, respectively), while $P$ and $Q$ are both connected.

Problem 13a

No. By definition, a periodic set is the union of all possible shifts of its fundamental domain by vertical and horizontal vectors of integer length.

Problem 13b

Yes. Consider the following example:



Problem 14

To solve Problem 14a, note that for a triangle the notion “inside” can already be defined: a triangle is a convex polygon and thus the intersection of closed half-planes $S_1,S_2,S_3$. By this definition, the internal points are just those contained in the intersection of the half-planes $S_1,S_2,S_3$ without boundaries. Also, by this definition, a segment connecting an internal point with an external one meets the boundary of one of the half-planes, and thus meets one of the edges of the triangle.

Problem 14a

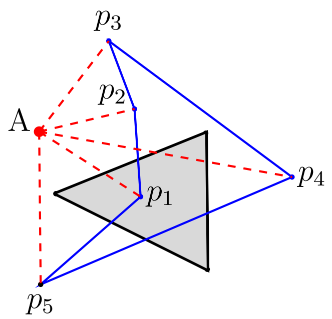
Consider one of the triangles as a convex polygon, and the other — as a closed polygonal chain. The remark above implies that this polygonal chain enters inside of the first triangle the same number of times as it leaves it, therefore, the number of times the polygonal chain meets the boundary of the triangle is even.

Problem 14b

Choose a point $A$ and connect it by segments with each vertex of the given polygonal chain. Obviously, one can choose $A$ such that none of those segments contains any of the vertices of the triangle $T$, and any two of those segments are not contained in the same line. Then, the edges of the polygonal chain are the bases of some triangles $T_1,\dots,T_n$ with the common apex at $A$, and none of the triangles $T_i$ contains the vertices of $T$. Thus, Problem 14a implies that the number of the intersection points of the triangle $T$ with each of the triangles $T_i$ is even, and the sought number of the intersection points differs from it by an even number (namely, by twice the number of times the triangle $T$ meets the segments connecting $A$ with the vertices of the chain).

Problem 14c

In analogy with Problem 14b, replace the first polygonal chain by a tuple of triangles. By Problem 14b, the number of the intersection points of the second polygonal chain with each of those triangles is even.




In the figure on the left-hand side, the edges $p_1$, $p_2$,..., $p_5$ of polygonal chain are the bases of the triangles $p_1Ap_2$, $p_2Ap_3$, $p_3Ap_4$, $p_4Ap_5$ $p_5Ap_1$ with the common vertex at the new point $A$. One can see that each of the segments $p_iA$ occurs exactly twice.

Problem 15a

Denote the new edge by $EF$. This edge is contained in some component $C$ of the complement to the graph $A$. We will denote by $A'$ the union of the graph $A$ with the edge $EF$ and will analyze the relation between the components of the complement to $A$ with the ones to $A'$.

Let $x$ and $y$ be points contained in the same component of the complement to $A$, that is, connected by a polygonal chain $P$ avoiding the graph $A$. If $P$ meets $EF$, then both $x$ and $y$ belong to the component $C$, otherwise, $x$ and $y$ belong to the same component of the complement to the graph $A'$. Hence all the components of the complement to $A$, except $C$, are also the components of the complement to $A'$.

In order to compute the number of components of the complement to $A'$ that are contained in $C$, on both sides of the edge $EF$ we shall construct the triangles $EFS$ and $EFT$ that have no common points with $A$ besides $E$ and $F$.

Let us prove that every point $x$ of the component $C$ can be connected with $S$ or $T$ by a polygonal chain avoiding the graph $A'$. This will imply that the component $C$ contains at most two components of the complement to the graph $A'$ – namely, the components of the points $S$ and $T$.

Since the point $x$ is contained in $C$, a polygonal chain $P$ connects $x$ to an interior point $G$ of the segment $EF$. Moreover, we can assume that $G$ is the only point of $P$ that the segment $EF$ contains. Otherwise we shall take the part of the polygonal chain $P$ from the point $x$ to the first point that $EF$ contains, and discard the rest of $P$.

Since the triangles $EFS$ and $EFT$ together cover a small neighborhood of the point $G$, the polygonal chain $P$ contains a nearby point $G'$ that does not belong to the segment $EF$, but does belong to one of the two triangles – say, to $EFS$.

As a result, the fragment of the polygonal chain $P$ between $x$ and $G'$, together with the segment $G'S$, forms a longer chain that connects $x$ and $S$, and avoids the graph $A'$.

Problem 15b

It remains to prove that the points $S$ and $T$ from the preceding solution are contained in different components of the complement to the graph $A'$. Indeed, otherwise they would be connected by a polygonal chain $P$ that avoids the graph $A'$. Assuming with no loss of generality, that the angles $S$ and $T$ in the triangles $EFS$ and $EFT$ are obtuse, we notice that the segment $ST$ is contained in the union of the triangles $EFS$ and $EFT$. So $ST$ avoids the graph $A$ and intersects $EF$ in one point $G$.

Thus the segment $ST$ together with the polygonal chain $P$ forms a closed non-self-intersecting polygonal chain $P'$, which intersect the graph $A'$ in a single point $G$.

On the other hand, since the graph $A$ is connected, it contains a polygonal chain $Q$ that connects the points $E$ and $F$. Extending $Q$ with the segment $EF$ to obtain a closed non-self-intersecting polygonal chain $Q'$, the closed non-self-intersecting polygonal chains $P'$ and $Q'$ do not contain each others' vertices and intersect at a single point $G$. This contradicts Problem **14**.

Quasiperiodic Sets

Problem 16a

Yes, it is. Denote the given finite set by $S$, and the periodic set with the fundamental domain being the fractional part $\{S\}$ — by $M$. Then, the intersection of $M$ with the given irrational line is exactly $S$.

Problem 16b

No, in analogy with the previous problem one can show that a segment is a quasiperiodic set.

Problem 16c

An example of such a set is the tuple of all points of the line that are on the right-hand side of the origin at an integer distance from it.
Assume the contrary: let $M$ be the intersection of the line $L$ and a periodic set with the fundamental polyhedron $F$. Since $M$ contains infinitely many points, so does the polyhedron $F$, therefore, it contains a segment $I$ that intersects the winding $L$.
But since $M$ does not contain a segment, the segment $I$ is not parallel to the line $L$. Thus, the density of the intersection points of the winding and the segment $I$ is nonzero. Therefore, $M$ has a non-empty intersection with any long enough segment of the line $L$, including an interval on the left-hand side of the origin. But $M$ contains no points on the left-hand side of the origin, which is a contradiction.

Remark. In fact, none of the rays contained in the given line is a quasiperiodic set. Try to show it.

Problem 17a

The answer is the area of the rectangle. Indeed, let the edge of the rectangle that is parallel to the given line be of the length $a$, and the edge perpendicular to it be of the length $b$. Then, each component of the set $Q$ is a length $a$ segment, while the density of those components equals the density of the winding with the perpendicular edge, that is, $b$. Hence the average length equals $ab$, that is, the area of $F$.

Problem 17b

The answer is the area of $F$. The proof consists in using the hint.

Problem 18a

The answer is the area of the fundamental domain, according to the previous problem. The area is equal to $2h - h^2$.

Problem 18b

The density of green tiles is equal to the length of the orthogonal projection of a green rectangle along the line $x + \sqrt{2}y = 0$. This length equals $\frac{1+\sqrt{2}}{\sqrt{3}}h$. Similarly, the density of yellow rectangles is the sum of densities of horizontal and vertical rectangles, i.e., $\frac{1+\sqrt{2}}{\sqrt{3}}$.
Notice that each component of $Q_h$ (except the first one started at the origin) consists of alternating yellow and green tiles, and the boundary tiles are always yellow. Hence the difference between the numbers of green and yellow tiles is equal to the number of components with maximal error of $\pm 3$. Therefore, the density of components of $Q_h$ is equal to the difference between the densities of yellow and green tiles, i.e., $\frac{1+\sqrt{2}}{\sqrt{3}}(1-h)$.

## Euler characteristic

Problem 19a

Denote the given figure by $P$. It is a polyhedron since it has the following cell decomposition: the trapezoid, with vertices $(0,0)$, $(1,0)$, $(1,-1)$, $(-1,-1)$, all its edges and vertices, the interval

with endpoints $(1,0)$ and $(1,1)$, its endpoints, and also the cells symmetric to the ones listed above in the line $y=x$.
Assume now that there exists the cell decomposition of $P$ containing the given triangle. Then the side $I$ of this triangle with endpoints $(1,-1)$ and $(-1,1)$ is also a cell. On the other hand, the triangle with vertices $(0,0)$, $(0,1)$ and $(-1,1)$ should contain at least one 2-dimensional cell with edge $J$ which lies inside of $I$. Therefore, we have found two different intersecting cells $I$ and $J$. A contradiction.

Problem 19b

By definition, the given polyhedron is a union of several polygons. Each of those polygons is an intersection of some half-planes. Denote by $L_1$, $L_2$, ..., $L_n$ the boundary lines of all these half-planes.
These lines define a cell decomposition of the whole plane as follows. Each 2-dimensional cell is an intersection of $n$ closed half-planes with boundary lines $L_1$, $L_2$, ..., $L_n$ (we take only non-empty such intersections). The 1- and 0-dimensional cells are defined in an obvious way.
Each cell in the constructed decomposition either is contained in $P$ or does not intersect $P$. Therefore, the set of cells contained in $P$ forms a cell decomposition of $P$.

Problem 20a

If the given polyhedron consists only of finite number of points, the equality is trivial.
Assume that the given polyhedron is a forest (i.e., a union of several trees). Indeed, any forest can be grown from a finite set of points by successive adjunction of leaves. At each step, one interval and one vertex is added. So the Euler characteristic remains the same, and the equality holds also for any forest.
Next, assume that the given polyhedron is a graph. Any graph can be obtained from a forest by adding some edges inside components. Due to Problem **15**, such operation adds one connected component to the complement of the graph and thus preserves Euler's equality.
Finally consider an obituary polyhedron. One can obtain this polyhedron by adding some polygons to a graph. Such operation increases the number of faces by 1 and decreases the number of components of the complement by 1 as well. So Euler's equality preserves.

Problem 20b follows immediately from Problem **20a**: the right-hand side of Euler's equality does not depend on a cell decomposition.

Problem 20d also follows immediately from Problem **20a** since the Euler characteristic of a closed non-self-intersecting polygonal chain is equal to 0.

Problem 20c is a direct consequence of the definition of the Euler characteristic, after one presents a cell decomposition of $P \cup Q$ such that those cells also form cell decompositions of polyhedra $P$, $Q$ and $P \cap Q$. A decomposition satisfying this condition always exists, e.g., it can be constructed using the same ideas as in the solution of Problem **19b**.

Problem 21

Our stroll to open problems is finished. Now it is your turn to share with us your progress and ideas concerning these problems. We will be glad to proceed our collaboration even after the Summer Conference. We are always open to discussions.

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