The Nagel, Gergonne, and Feuerbach points and their properties

A.Yakubov, A.Zaykov, M.Didin, P.Kozhevnikov, D.Krekov, A.Zaslavsky, O..Zaslavsky

2. Main problems

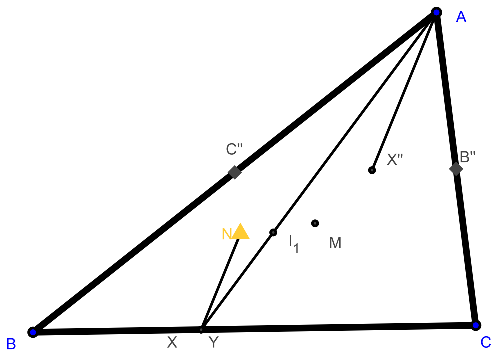
14. Consider the polar transformation wrt the incircle. Point C is the pole of line A_0B_0, and the infinite point of AB is the pole of line C_0I. Therefore $S = A_0B_0 \cap C_0I$ is the pole of line n passing through C and parallel to AB. Then the quadruple of lines CA_0, CB_0, CS and n is harmonic. These lines meet AB at A, B, infinite point and the midpoint of the side (because the cross-ratio is equal to -1), hence C, S and C'' are collinear.

15. Take the composition of the inversion with center A and radius $\sqrt{AB \cdot AC}$ and the reflection about the bisector of angle BAC. It maps B to C and vice versa. Hence the image of line BC is the circumcircle of triangle ABC. The image of the incircle touches the rays AB and AC and touches the circle ABC externally at point X which is the image of A_0. Therefore G_2 lies on AX. Since A is the external homothety center of the incircle and its image, and X is the internal homothety center of the circumcircle and the image of the incircle, we obtain that the internal homothety center of the incircle and the circumcircle lies on the line AX coinciding with AG_2. Similarly it lies on BG_2 and CG_2. Hence this center coincide with G_2. Similarly the external homothety center coincide with N_2.

16. It is known that the isogonal and the isotomic conjugations map lines to circumconics and vice versa. Therefore their composition maps lines to lines, i.e this transformation is projective. It maps N_2 to G, G_2 to N, I to I_1, and O to H_1. By the assertion of problem 15, the quadruple O, I, G_2, N_2 is harmonic. Therefore the quadruple H_1, I_1, N, G is also harmonic. The prove for the remaining quadruples is similar.

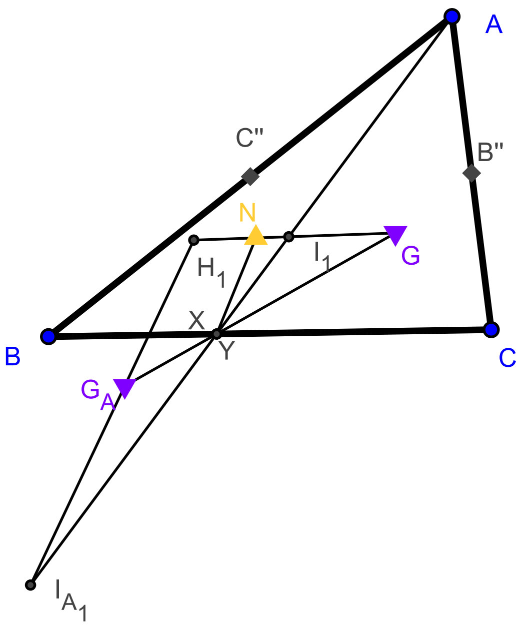
17. Let X be the common point of NN_A and BC, Y be the common point of $I_{A_1}I_1$ and BC. Since I_1 and I_{A_1} are isotomically conjugated to I and I_A respectively, we obtain that $\frac{\overline{BY}}{\overline{YC}} = \frac{AC}{AB}$. Consider the homothety with center M and coefficient $-\frac{1}{2}$. By the assertion of problem 7 it maps the line NN_A to II_A, i.e the bisector of angle A. Also it maps B and C to B'' and C'' respectively, and X to X''. By the bisector

property $\frac{\overline{B''X''}}{\overline{X''C''}} = \frac{B''A}{C''A} = \frac{CA}{BA} = \frac{\overline{BY}}{\overline{YC}}$. Also $\frac{\overline{B''X''}}{\overline{X''C''}} = \frac{\overline{BX}}{\overline{XC}}$. Hence X and Y coincide. Thus $I_{A_1}I_1$, NN_A, BC concur.



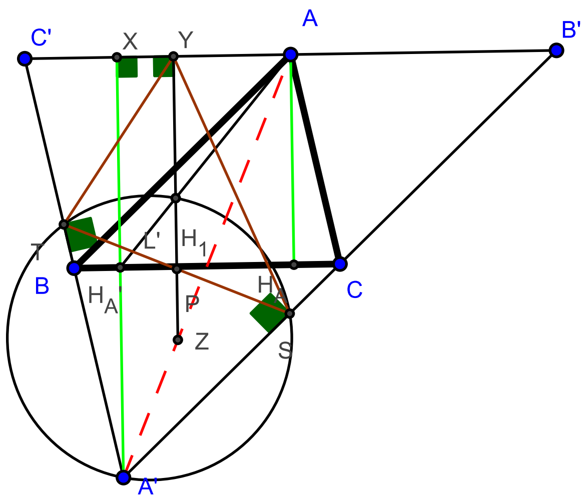
Similarly $I_{B_1}I_{C_1}$, N_BN_C, BC concur.

18. The points I_1, H_1, N, G are collinear; and I_{A_1}, H_1, N_A, G_A are collinear. Also the quadruples I_1, H_1, N, G and I_{A_1}, H_1, N_A, G_A are harmonic. Therefore the lines $I_{A_1}I_1$, NN_A, GG_A concur. Using the assertion of problem 17 we obtain the required one.

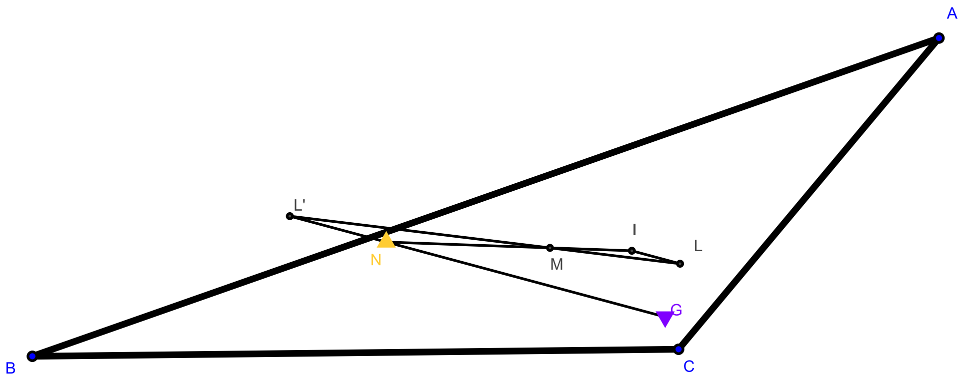


19. By the assertion of the problem 16 all indicated lines pass through H_1. Thus we have to prove that H_1 and L' coincide. Prove that

AH_1 passes through L'. Let X and Y be the projections of A' and L' respectively to $B'C'$, and let Z be the reflection of Y about L'. The line AH_1 passes through the reflection of H_A about A''. Note that the reflection about A'' maps the altitude from A to the line $A'X$. The line BC bisects $A'X$. Therefore AH_1 passes through the midpoint of $A'X$. The quadrilateral $ZYXA'$ is a trapezoid, and L' is the midpoint of its base YZ. Hence AH' passes through the midpoint of $A'X$. To obtain that AH' passes through L', prove that A is the common point of lateral sidelines, i.e. A, Z, A' are collinear. Let S and T be the projections of L' to $A'B'$ and $A'C'$ respectively. The triangle STY is the pedal triangle of L' wrt $A'B'C'$. Since L' is the Lemoine point of $\triangle A'B'C'$, we obtain that L' is the centroid of its pedal triangle. Let P be the common point of ST and YL'. Then YP is a median of triangle YST. Therefore $\frac{\overline{YL'}}{\overline{L'P}} = 2$, $\overline{YL'} = \overline{L'Z}$. From this P is the midpoint of $L'Z$. The common point of ST and $L'Z$ bisects these segments. Hence $SL'TZ$ is a parallelogram. Since $SL' \perp SA'$ and $TL' \perp TA'$, we obtain that $SZ \perp A'T$ and $TZ \perp A'S$, i.e. Z is the orthocenter of triangle $SA'T$, and L' is opposite to A' on its circumcircle. Hence the lines $A'L'$ and $A'Z$ are symmetric wrt the bisector of angle $B'A'C'$, i.e. $A'L'$ is a symedian of $\triangle A'B'C'$, and $A'Z$ is its median. Since A is the midpoint of $B'C'$, we obtain that A', Z, A are collinear.



20. The homothety with center M and coefficient $-\frac{1}{2}$ maps N and L' to I and L respectively. Thus the lines IL and NL' are parallel. But by the assertion of the problem 19 the points L', N, G are collinear.



Similarly the lines N_AG_A and I_AL, N_BG_B and I_BL, N_CG_C and I_CL are parallel.

21. By the assertion of the problem 18 the perspective axes of triangles $N_AN_BN_C$ and ABC, $G_AG_BG_C$ and ABC coincide with the line through $AB \cap I_{A_1}I_{B_1}$, $AC \cap I_{A_1}I_{C_1}$, $BC \cap I_{B_1}I_{C_1}$.
The perpendicularity can be proved by the Sondat theorem (problem 23) or the properties of trilinear polars (problems 25-29).

22. The triangle $N_AN_BN_C$ is homothetic to $I_AI_BI_C$ with center M. Hence $N_AN_B \parallel I_AI_B$, i.e. $N_AN_B \perp CI$. Similarly $N_AN_C \perp BI$; $N_BN_C \perp AI$. Therefore I is the orthology center of triangles $N_AN_BN_C$ and ABC.

23. The triangles $N_AN_BN_C$ and ABC are perspective with center G. Thus it is sufficiently to prove that I is one of their orthology centers which follows from the assertion of problem 22.

3. Additional problems. Hints.

24. The trypolar of G coincide with its polar wrt the incircle.

25. It is sufficiently to consider a regular triangle.

26. Follows from the Ceva and the Menelaus theorem.

27. A projective transformation reduces this problem to the problem 25.

31. It is sufficiently to consider a regular triangle.

31', 32. Both assertions are equivalent to the following one: the tangent to the incircle at the Feuerbach point touches also *the Steiner inellipse*, which touches the sides of the triangle at its midpoints (see. [1]).

33. By the assertion of the problem 16 the Nagel point lies on the *Feuerbach hyperbola* (see. [2]).

References

[1] http://www.jcgeometry.org/Articles/Volume1/JCG2012V1pp23-31.pdf
[2] A.V.Akopyan, A.A.Zaslavsky. Geometry of conics. AMS, 2007.