The Nagel, Gergonne, and Feuerbach points and their properties

1. Introductionary problems

A.Jakubov, M.Didin, P.Kozhevnikov, D.Krekov, A.Zaslavsky, O.Zaslavsky

0. *Ceva theorem.* Let A', B', C' be points on the sidelines of a triangle ABC. Prove that AA', BB', CC' are conurrent or parallel iff $BA' \cdot CB' \cdot AC' = -CA' \cdot AB' \cdot BC'$ (here we consider directed segments).

Let ABC be a triangle. Let the inscribed circle touches the sides BC, CA, AB at points A_0, B_0, C_0, respectively.

1. Prove that[1] AA_0, BB_0, CC_0 have a common point G.

The point G from the previous problem is called *the Gergonne point* of the triangle. Consider the excircle touching the side BC. Let this excircle touches AB, AC, and BC at C_A, B_A, and A_A, respectively.

2. AA_A,BB_A, and CC_A have a common point G_A.

Let us call the point G_A from the previous problem *the external Gergonne point* corresponding to A. Similarly define points A_B, B_B, C_B, G_B, A_C, B_C, C_C, G_C.

Introduce some standart notation. Let I be the incenter, and I_A, I_B, I_C be the centers of the excircles. Let Be, Be_A, Be_B, Be_C be the centers of the circles $I_AI_BI_C$, II_BI_C, I_AII_C, I_AI_BI, respectively (Be is called *the Bevan point*, let us call Be_A, Be_B, Be_C *the external Bevan points*). Let O, H, M be the circumcenter, the orthocenter, the centroid of ABC, respectively. Let A'',B'',C'' be the midpoints of BC, CA, AB, respectively. Let A',B',C' be reflections of A, B, C in A'',B'',C'', respectively, let H' be the orthocenter of $A'B'C'$.

3. O, M, H are collinear, and $\frac{\overline{OM}}{\overline{MH}} = \frac{1}{2}$.

The line from the previous problem is called *the Euler line* of the given triangle.

3'. $\frac{\overline{H'M}}{\overline{H'O}} = \frac{4}{3}$.

[1] in the next problems the words "Prove that" will be omitted

4. Let X_A, X_B, X_C be points on the sides BC, CA, AB, respectively, so that AX_A, BX_B, CX_C intersect at point X. Let Y_A, Y_B, Y_C be reflections of X_A, X_B, X_C in A'', B'', C'', respectively. AY_A, BY_B, CY_C have a common point Y.

Points X and Y from the previous problem are called *isotomically conjugates* with respect to ABC.

5. AA_A, BB_B, CC_C have a common point N which is isotomically conjugate to the Gergonne point.

The point from the previous problem is called *the Nagel point.*

6. Lines AA_0, BB_C, CC_B have a common point N_A that is isotomically conjugate to G_A.

The point from the previous problem is called *the external Nagel point* corresponding to A. Points N_B and N_C defined similarly.

To each direction put into correspondence a point at infinity (assume that this point of infinity lies on each line of this direction). Triangles $X_1Y_1Z_1$ and $X_2Y_2Z_2$ are called *perspective* if lines X_1X_2, Y_1Y_2, and Z_1Z_2 are concurrent. In this case the common point of X_1X_2, Y_1Y_2, and Z_1Z_2 is called *the perspector* of triangles $X_1Y_1Z_1$ and $X_2Y_2Z_2$ (the perspector could be a point at infinity).

7. M cuts each of segments IN, I_AN_A, I_BN_B, I_CN_C at ratio 1:2.
Corollary. Triangles $I_AI_BI_C$ and $N_AN_BN_C$ are perspective with M as a perspector.

8. O is the midpiont of segments BeI, Be_AI_A, Be_BI_B, Be_CI_C.
Corollary. Triangles $I_AI_BI_C$ and $Be_ABe_BBe_C$ are perspective with O as a perspector.

9. Be,Be_A,Be_B,Be_C are the midpoints of the segments $H'N$,$H'N_A$,$H'N_B$,$H'N_C$, respectively.
Corollary. Triangles $Be_ABe_BBe_C$ and $N_AN_BN_C$ are homothetic (hence perspective) with center H' and ratio 2.

10. Lines C_CG_A, A_AG_C, BG are concurrent.

11. Lines A_BC_0, G_BG, BI, C_BA_0 are concurrent.

11'. Lines A_AC_C, G_AG_C, I_AI_C, A_CC_A are concurrent (as well as analogous triples of lines).

12. $I_AG_A \cap I_CG_C$, Be, N are collinear.

12'. $I_AG_A \cap I_CG_C$, Be_B, N_B are collinear.

13. I_AG_A, I_BG_B, I_CG_C, and IG, meet at H'.
Corollary: Triangles $I_AI_BI_C$ and $G_AG_BG_C$ are perspective with perspector H'.