The Nagel, Gergonne, and Feuerbach points and its properties

1. Introductory problems. Solutions

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1. Follows from the Ceva theorem.

2. Follows from the Ceva theorem.

3. Consider the homothety with center M and coefficient -2.

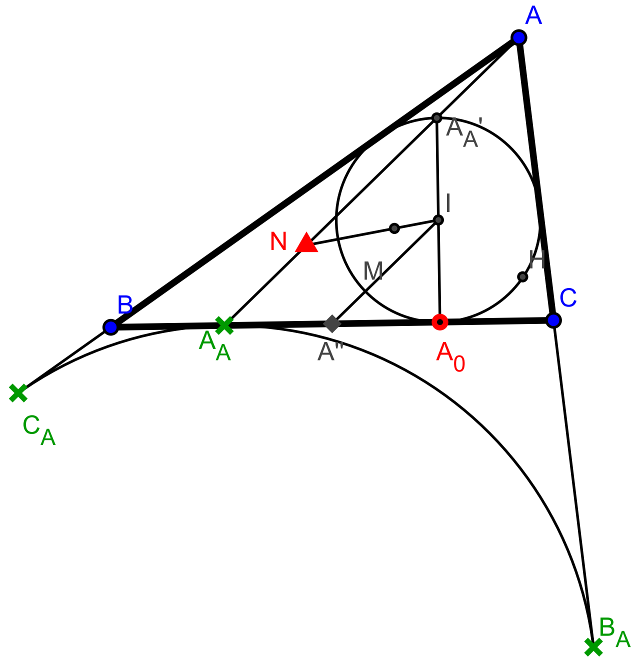
3'. Use the assertion of problem 3 and the homothety with center M and coefficient -2.

4. Follows from the Ceva theorem.

5. Follows from the Ceva theorem.

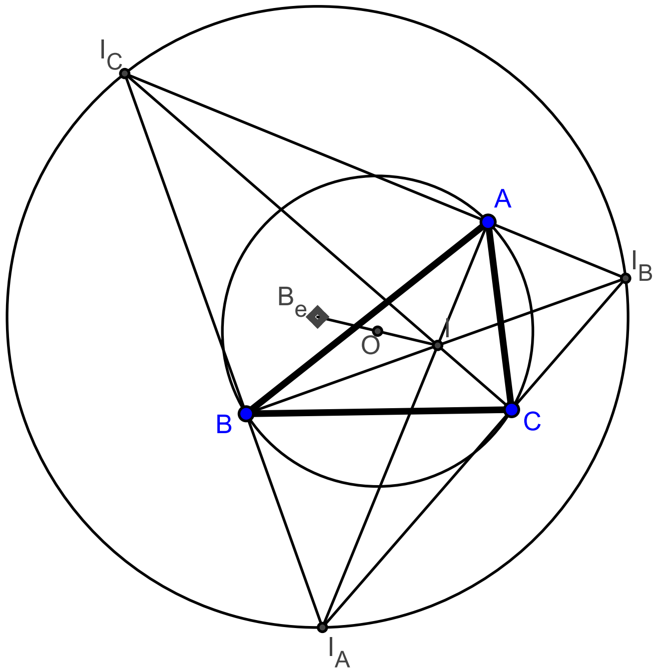
6. Follows from the Ceva theorem.

7. It is kn0wn that $BA_0 = CA_A$, hence A'' is the midpoint of segment A_0A_A. By the definition of isotimic conjugation N lies on AA_A. Take the homothety with center A transforming the excircle touching the side BC to the incircle. Let it map A_A to A'_A. The tangent to the incircle at A'_A is parallel to eh tangent to the excircle at A_A coinciding with the line BC. Also it is clear that A'_A is distinct from A_0. Therefore A_0 and A'_A are opposite and I is the midpoint of $A_0A'_A$. Hence $A''I$ is a medial line of $\triangle A'_AA_0A_A$. Hence $AA_A \| A''I$. Similarly $BB_B \| B''I$; $CC_C \| C''I$. Therefore the homothety with center M and coefficient -2 transforming $\triangle A''B''C''$ to $\triangle ABC$ maps I to N. Thus M divides IN in ratio 1:2.

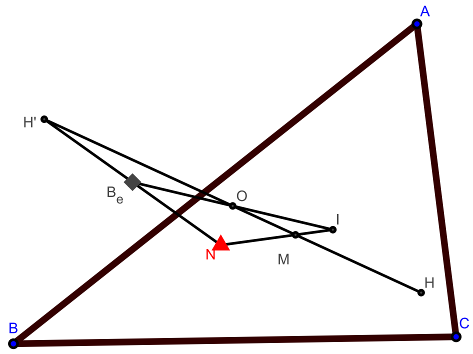


The assertion for the segments I_AN_A, I_BN_B, I_CN_C may be proved similarly.

8. The internal and the external bisectors of an arbitrary angle are perpendicular. Therefore I is the orthocenter of triangle $I_AI_BI_C$, and I_A, I_B, I_C are the orthocenters of $\triangle II_BI_C$, $\triangle I_AII_C$, $\triangle I_AI_BI$ respectively. The points A, B, C are the feet of altitudes in these four triangles, Therefore O is the center of their common nine-point-circle. The center of NPC bisects the segment between the circumcenter and the orthocenter. From this we obtain the required assertion.



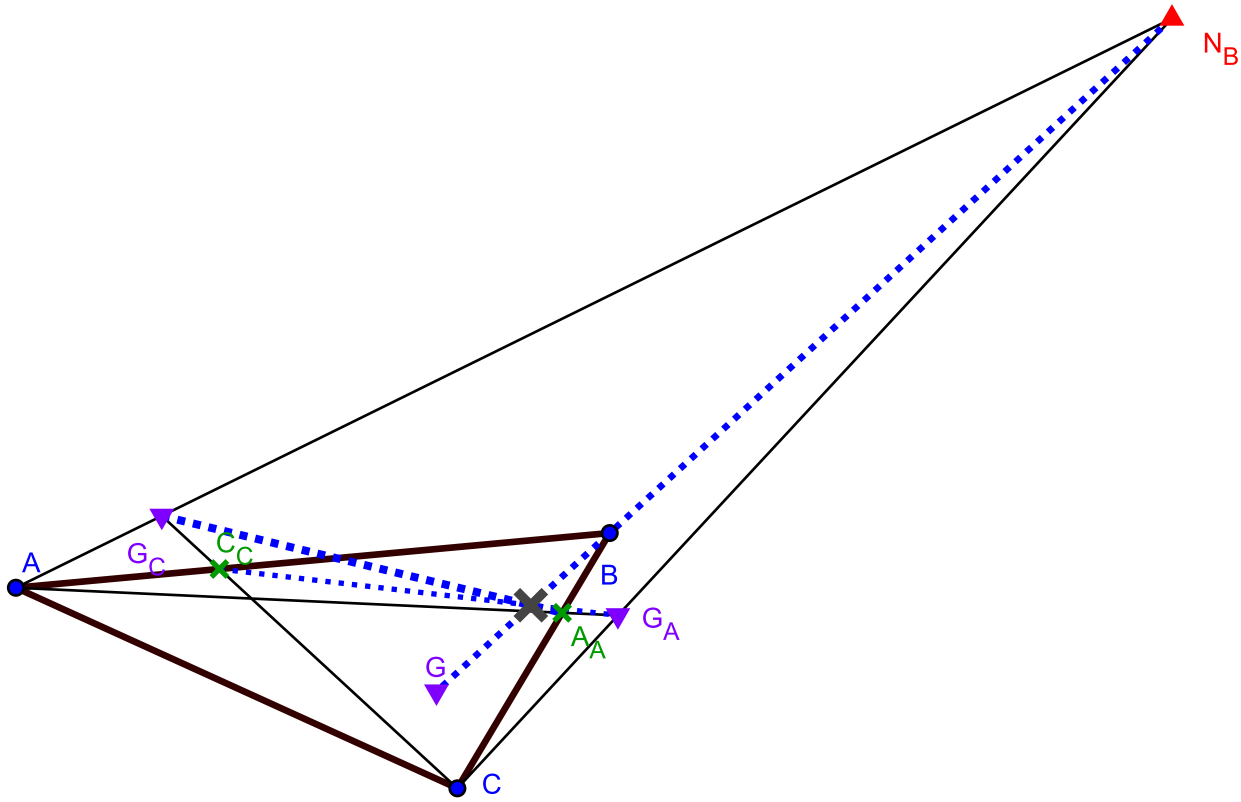
9. We proved above that $\frac{\overline{H'M}}{\overline{H'O}} = \frac{4}{3}$; $\frac{\overline{NI}}{\overline{NM}} = \frac{3}{2}$; $\frac{\overline{BeO}}{\overline{BeI}} = \frac{1}{2}$. Using the Menelaus theorem to $\triangle OMI$, we obtain that H',N, Be are collinear. Now using the Menelaus theorem to the line MO and the triangle $IBeN$ we obtain that $\frac{\overline{H'Be}}{\overline{H'N}} = \frac{\overline{OBe}}{\overline{OI}} \cdot \frac{\overline{MI}}{\overline{MN}}$
$\frac{\overline{OBe}}{\overline{OI}} = -1$; $\frac{\overline{MI}}{\overline{MN}} = -\frac{1}{2}$. Hence $\frac{\overline{H'Be}}{\overline{H'N}} = \frac{1}{2}$, i.e. Be is the midpoint of $H'N$.



The assertion for Be_A, Be_B, Be_C may be proved similarly.

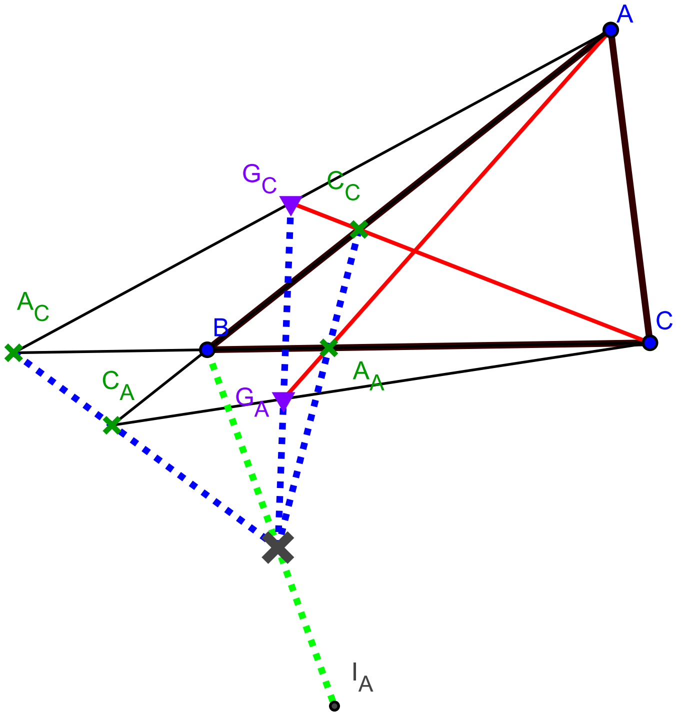
Remark. Let l be the line passing through H', Be, N. Define the lines l_A, l_B, l_C similarly. If $\triangle ABC$ is not isosceles, then the lines l, l_A, l_B, l_C are distinct. In fact, suppose for example that l and l_A coincide. Consider the homothety with center M and coefficient $-\frac{1}{2}$. It maps N, N_A, N_B, N_C to I, I_A, I_B, I_C respectively, also it maps H' to H. If l and l_A coincide, then I_A,I, H are collinear. Hence these points lie on the bisector of angle BAC. This line is also the altitude because it passes through H, therefore ABC is isosceles — contradictory.

10. The points A, A_A, G_A are collinear. Similarly C, C_C, G_C are collinear. By the Pappus theorem $AC_C \cap CA_A$, $C_CG_A \cap A_AG_C$, $AG_C \cap CG_A$ are collinear. Since $AC_C \cap CA_A = B$ we have to prove that $AG_C \cap CG_A$ lies on BG. But AG_C passes through A_C, CG_A passes through C_A, and BG passes through B_0. Therefore the lines AG_C, CG_A, BG pass through N_B.


N_B
G_C
C_C
A
B
G_A
A_A
G
C

11'. Use the Pappus theorem to A, A_A, G_A and C, G_C, C_C. We obtain that $AC_C \cap CG_A$, $C_CA_A \cap G_AG_C$, $AG_C \cap CA_A$ are collinear. Note that $AC_C \cap CG_A = C_A$; $AG_C \cap CA_A = A_C$. Therefore the lines C_CA_A, C_AA_C, G_AG_C concur.

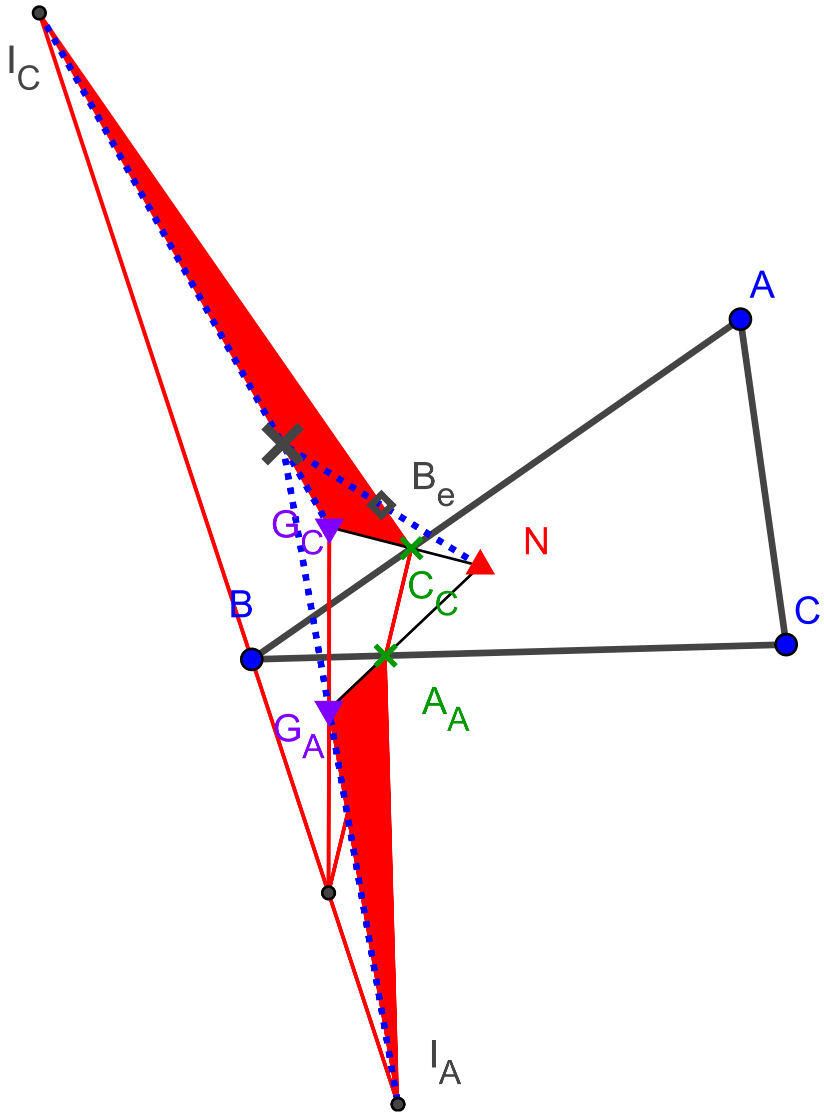
The segments BC_A and BA_A are congruent as two tangents to the excircle touching BC. Hence C_A and A_A wrt the external bisector of angle B. Similarly C_C and A_C are symmetric wrt the external bisector of angle B. Therefore the lines A_AC_C and A_CC_A meet on I_AI_C. Hence the lines C_CA_A, C_AA_C, G_AG_C, I_AI_C concur.



11. Similarly we can prove for example that A_BC_0, C_BA_0, G_BG, BI concur, applying the Pappus theorem to A, G_B, B_A and C, C_0, G.

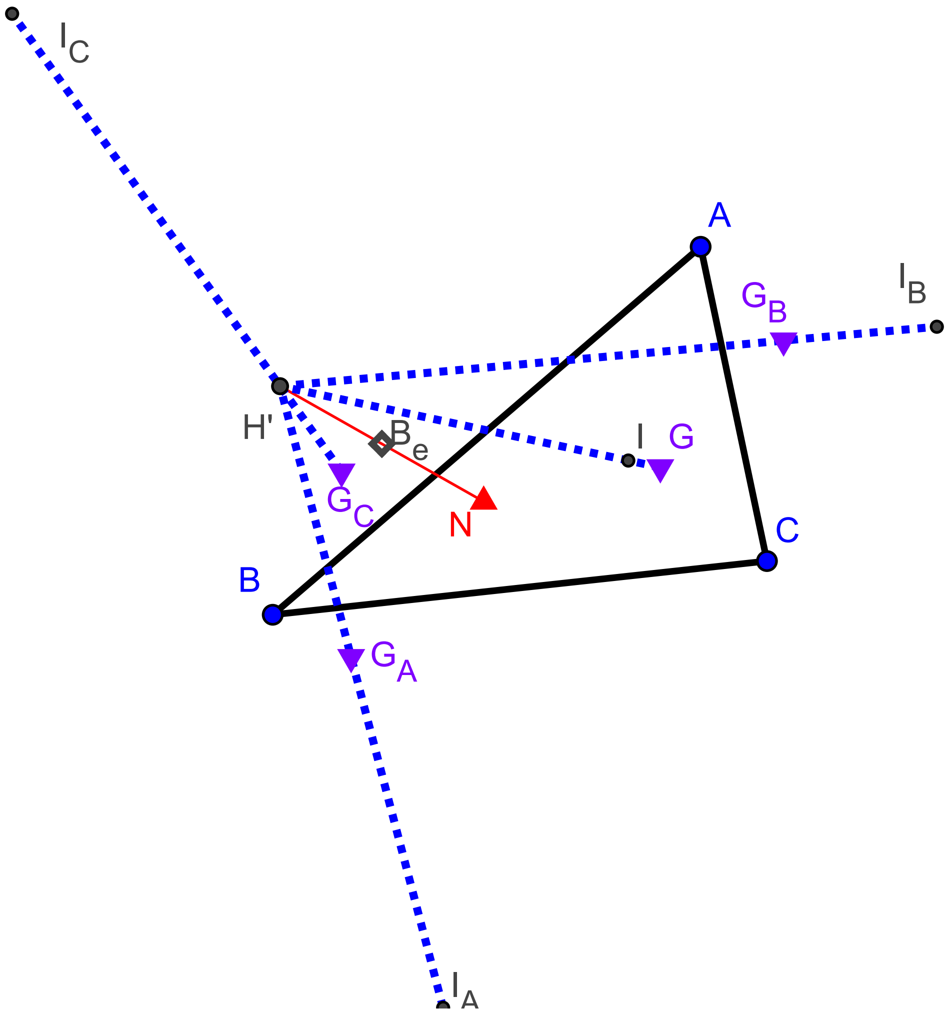
12. By the assertion of problem 11 the lines A_AC_C, G_AG_C, I_AI_C concur. Therefore the triangles $A_AG_AI_A$ and $C_CG_CI_C$ are perspective. By the Desargues theorem the points $A_AG_A \cap C_CG_C$, $A_AI_A \cap C_CI_C$, $G_AI_A \cap G_CI_C$ are collinear. Now $A_AG_A \cap C_CG_C = N$, the triangle ABC is the orthotriangle of $\triangle I_AI_BI_C$. Since $I_CC_C \perp AB$ (the sideline of the orthotriangle), we obtain that I_CC_C passes through the circumcenter of $\triangle I_AI_BI_C$. Similarly I_AA_A passes through this circumcenter. Thus $A_AI_A \cap C_CI_C = Be$. Hence

$I_A G_A \cap I_C G_C$, Be, N are collinear.



12'. Using the Desargues theorem to $\triangle C_A G_A I_A$ and $\triangle A_C G_C I_C$, we obtain thah $I_A G_A \cap I_C G_C$, Be_B, N_B are collinear.

13. By the assertion of problem 12, $I_A G_A \cap I_C G_C$ lies on l. Similarly $I_A G_A \cap I_C G_C$ lies on l_B. If $\triangle ABC$ is not isosceles l and l_B are distinct. By the assertion of problem 9 both lines pass through H'. Therefore $I_A G_A \cap I_C G_C = H'$. Similarly IG and $I_B G_B$ pass through H'. It is clear that the assertion is also correct for isosceles triangles.



I_C
A
G_B
I_B
H'
B_e
I
G
G_C
N
C
B
G_A
I_A