What is a color of my hat?

The following problem is well known, but if you miss it before, please, consider it as a challenge. We will discuss this problem after the opening of the conference, it will not affect on results of the competition. The object in the problem has 4 states only!

Intellectual CHALLENGE: the number 4 against milliards of neurons of your brain!

Black or white hats are placed on your and on mine heads. You see my hat, I see your hat, but none of us sees the hat on his own head. Each of us (without any sort of communications) must try to guess the color of his hat. When a signal is given each of us simultaneously says one word only: «black» or «white». We will win if and only if at least one of us has guessed correctly. Before this test we hold a consultation. How should we act in order to win in all possible situations?

1 Several problems about sages

Several sages take part in the following TEST. There are a lot of hats of k different colors. The emcee places hats on the sages' heads. Each sage sees the hats of all other sages and does not see his own hat. The sages do not communicate. When a signal is given they simultaneously name one of colors. The sages win if and only if at least one of them has guessed correctly.

The sages hold a CONSULTATION before the test in order to coordinate their strategy during the test. Repeat that the only form of action is allowed during the test: to say one word just after a signal (independently of other sages). The strategy of sages should be deterministic, i.e. each sage decision is determined uniquely by the hats of other sages.

1.1. There are hats of n colors and n sages. Prove that the sages win.

1.2. There are hats of three colors and n sages are arranged in a line so that each sage can see only his neighbours (the leftmost and rightmost sage see one neighbour). Prove that the sages loose.

a) $n = 3$; b) $n = 4$; c) n is arbitrary.

1.3. There are hats of k colors and $10k$ sages (everybody sees all others). Prove that 10 sages can guess their colors correctly, but in general situation none 11 sages guess their colors correctly.

1.4. There are $4k - 1$ sages, $2k$ black hats and $2k$ white hats. The emcee hides one hat and all other hats place on the sages' heads. What maximal number of sages can guess their color correctly?

1.5. Four sages stand around a non-transparent baobab. The hats are of three colors. A sage sees only his two neighbours. How should they act to win?

1.6. Sages has hats of two colors. It is allowed to say «pass» during guessing, that means that a sage do not make a guess. The sages win if and only if at least one of them has guessed correctly and none of them has guessed incorrectly. We assume that all hats placements have equal probabilities and the sages strategy is deterministic as in previous problems. It is clear that now the sages can not to guarantee 100 % victory. For example a strategy «Sage A always says "black" and all others say "pass"» wins in one half of all possible cases. We call a strategy *optimal* if it wins the most number of all possible cases.

a) Find a strategy that wins in more than 50 % cases.

b) Find an optimal strategy and prove that it is optimal.

2 Sages on a non oriented graph

We will consider the following general problem. Let G be a non oriented graph and let sages live at its vertices: one sage occupies one vertex. All the sages are familiar with each other and all of them know the whole placement of sages on the vertices of the graph. In particular, each sage understand in what vertex do he and his neighbours live. We will identify a vertex and the sage in it. During the test each sage sees only the hats of sages in the adjacent vertices. Other rules are the same: during the consultation the sages should choose a strategy that allows at least one of them to guess the color of his hat correctly.

We will use the following formalism. Let the colors of hats be numbered from 1 to k and let $\mathcal{C} = \{1, 2, \dots, k\}$. For each vertex v of G order the adjacent vertices by increasing of their numbers (denote by d the number of these vertices): $u_{n_1}, u_{n_2}, \dots u_{n_d}$. A strategy of the sage v is a function $f_v \colon \underbrace{\mathcal{C} \times \mathcal{C} \times \dots \times \mathcal{C}}_{d \text{ times}} \to \mathcal{C}$. The sages choose these functions on the consultation. During the test a sage v calculates $f_v(c_1, c_2, \dots, c_d)$, where $c_i \in \mathcal{C}$ is a color of the sage in the vertex v_{n_i}.

The problem 1.1 shows that if the graph G contains a k-clique, then at least one sage can guess the color of his hat correctly. But if the graph does not contain a k-clique, the question becomes non trivial.

2.1. Let $k \geqslant 3$. Prove that for 4-vertex graph “chicken feet” the sages loose.

2.2. . Prove that for an arbitrary tree the sages loose ($k \geqslant 3$).

Now let n sages live at the vertices of a cycle, $k = 3$. Let V be a 3-element set of hats colors. Denote by $V_i = V$ the set of colors of hats that will be placed on the head of the i-th sage. Assume that the sages have chosen a strategy. That means that i-th sage has a function $f_i \colon V_{i-1} \times V_{i+1} \to V_i$ (we use cyclical numbering). A sequence of colors abc, where $a \in V_{i-1}$, $b \in V_i$, $c \in V_{i+1}$, is called a short *disproving chain* if $b \neq f_i(a, c)$. A long sequence $S = s_1 s_2 \dots s_m$, where $s_1 \in V_\ell$, $s_2 \in V_{\ell+1}$, …, $s_m \in V_{\ell+m-1}$, is called a *disproving chain* if each its 3-element consecutive subsequence is a short disproving chain. For every disproving chain S denote by $\ell_+(S)$ the number of continuations of this sequence by one step to the right, i.e. the number of ways to choose a color $s_{m+1} \in V_{\ell+m}$ that gives us a longer disproving chain. Denote by $\ell_-(S)$ the analogous number of continuations by one step to the left.

2.3. Let n sages live at the vertices of a cycle, $k = 3$. Prove that if there exists a disproving chain $S = s_1 s_2 \dots s_m$, where $2 \leqslant m \leqslant n-1$, for which the inequality $\ell_-(s_1 s_2) + \ell_+(s_{m-1} s_m) \geqslant 5$ holds then the strategy of sages does not win.

2.4. Let n sages live at the vertices of a cycle, $k = 3$. Let the sages choose a winning strategy. Prove that for each sage i and any pair of colors $a \in V_{i-1}$, $b \in V_i$ the equality $\ell_-(ab) + \ell_+(ab) = 4$ holds.

2.5. Prove that for $k = 3$ the sages win on the graph “a cycle of 3n vertices”.

2.6. Prove that for $k = 3$ the sages loose on the graph “a cycle of n vertices”, where n is not divisible by 3 and $n \neq 4$.

The following problems show that the sages can win in graphs without big cliques.

2.7. Prove that for any number of hats k there exists a bipartite graph for which the sages win.

2.8. Let G be a graph for which the sages win when the number of colors equals q. Let K_r be a complete graph on r vertices (we know that the sages win on this graph when the number of colors equals r). Construct a new "big" graph $\widetilde{G}$. For this replace each vertex of the graph G by a copy of graph K_r. If the two vertices were adjacent in G draw the edges between all pairs of vertices in the corresponding copies of K_r. The obtained graph is $\widetilde{G}$.

Prove that the sages win on the graph $\widetilde{G}$ when the number of colors equals $k = qr$.

2.9. Prove that for $k = 3m$ there exists a graph with $4m$ vertices and maximal clique of size at most $2m$, for which the sages win.

3 *Sages on an oriented graph*

Now let the sages live at the vertices of oriented graph; the sage A sees the sage B if and only if the graph contains an oriented edge AB.

3.1. Prove that the sages win on the graph "oriented cycle of n edges" ($k = 2$).

3.2. Denote by c the maximal number of vertex dijoint cycles in a graph. Prove that there exist graphs for which more than c sages can guess the colors correctly ($k = 2$).

3.3. Let a be the minimum number of vertices whose removal makes the graph acyclic. Prove that at most a sages can guess the colors correctly ($k = 2$).

3.4. An oriented graph G is called *semibipartite* if its vertex set can be split onto two parts L and R so that there no edges between vertices of L, and R is acyclic (the edges from L to R and from R to L are not forbidden).

Let the sages have hats of k colors and s be an arbitrary non negative integer. Prove that the sages loose on a semibipartite graph if $|L| = k - 2$, $|R| = s$.

After semifinal

Variations of previous topics

2.10. There are three sages A, B, C, each sees each other, except that the sage A does not see the sage B; $k = 3$. Prove that sages loose.

2.11. Four sages stand around a non-transparent baobab. The hats are of three colors. A sage sees only two his neighbours, except one sage who sees only one his neighbour. Can the sages win?

Let n sages stand at the vertices of a cycle, $k = 3$. Suppose that sages chose a winning strategy. The pair of colors ab, where $a \in V_i$, $b \in V_{i+1}$, will be called a *left* pair, if $\ell_-(ab) = 1$, will be called a *right* pair, if $\ell_-(ab) = 3$, and will be called an *inert* pair, if $\ell_-(ab) = 2$.

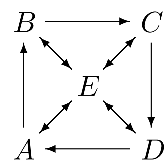
2.12. Prove that the number of left pairs equals the number of right pair among all the pairs ab, such that $a \in V_i$, $b \in V_{i+1}$.

2.13. Let ab be a right pair of colors, $a \in V_i$, $b \in V_{i+1}$. Prove that among the pairs of colors c_1a, c_2a, c_3a, such that $\{c_1, c_2, c_3\} = V_{i-1}$, there is exactly one left pair, exactly one right pair and exactly one inert pair.

2.14. The same setting as in Problem 1.4, but there are $mk - 1$ sages and hats of k colors, m hats of each color, either m is even, or k is odd (possibly both is true). The emcee hides one hat. Prove that the maximal number of sages who can guess their color correctly is $\frac{1}{2}(mk + m - 2)$.

2.15. The sages stand in two lines: n sages in the first line and n^n sages in the second lines. They have hats of $(n+1)$ colors. The sages see only sages standing in the other line. олько тех, кто стоит в другой шеренге. When a signal is given, each of the sages simultaneously names a color. Prove that the sages can act in such a way that at least one guesses.

3.5. What maximal number of sages can guess on the following graph ($k = 2$)?



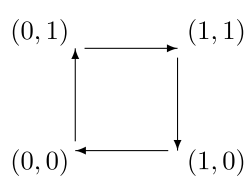
4 *Hypercube.*

By an n-dimensional hypercube we mean a graph, such that its vertices are numbered by sequences of n zeroes and ones. Two vertices are joined by an edge if and only if their numbers differ only in one digit.

4.1. Prove algebraically that 32 sages, standing in the vertices of a 5-dimensional hypercube, can win ($k = 3$).

Suppose that there are n sages, $k = 2$. Let us denote the colors of hats by one and zero. Let us fix a strategy of the sages. Consider an n-dimensional hypercube and "encode" this strategy with it in the following way. Since the vertices of the hypercube correspond to sequences of n zeros and ones, we relate the ith sage and the ith element of this sequence. Consider the example for $n = 5$, $i = 2$. Suppose that the ith sage sees the colors of hats of the other sages, for instance, 1, *, 0, 1, 1 (the star means that the ith, i.e., the second sage, does not see his own hat color). There are two vertices of the hypercube with such coordinates, namely, $(1, 0, 0, 1, 1)$ and $(1, 1, 0, 1, 1)$, moreover, these vertices are joined by an edge. The strategy of the ith sage is to choose among these two vertices. Let us put an arrow on the corresponding edge, its tail being a non-chosen vertex, its head being a chosen vertex. Putting such arrows on all the edges, we get an illustration of the strategy.

For example, the strategy of sages from the Intellectual CHALLENGE can be described by the following orientation of the 2-dimensional hypercube:



4.2. Suppose that there are n sages, hats can be red or blue. Each sage sees each other. As we know from Problem 1.3, $[n/2]$ sages can guess their color correctly. Suppose that there exists a "balanced with respect to colors" strategy: such that for every distribution of hats, if there are r red and b blue hats, it is true that at least $[r/2]$ sages with red hats guess, and at least $[b/2]$ sages with blue hats guess.

4.3. Suppose that $2n$ sages use the optimal strategy, i.e., the strategy which leads to at least n guesses. Prove that this strategy is "unbiased" (with respect to one of the colors), namely: for every sage it is true, that if we consider all the distributions of hats, he says "red" in exactly half of the cases and "blue" in the other half of the cases according to his strategy.

Solutions

1.1. Let us label colors by residues modulo n. Every sage sees all the hat besides his own one. Let kth sage check the hypothesis “the sum of all the hats equals k modulo n. Then exactly one sage guesses.

1.2. This is a partial case of Problem 2.2.

1.3. [1, Theorem 2]

First we present a strategy for 10. Divide $10k$ sages into 10 groups of k sages each, and use Problem 1.1.

Assume there exists a strategy, that guarantees at least 11 correct guesses in each situation. Consider all k^{10k} ways to arrange colors to hats. Consider the k situations, that differ only in the color of the first hat. Since the strategy is deterministic, in all these situations the first sage will name the same color. Thus in these k situations the first sage will make only one correct guess. Dividing all k^{10k} situations into k^{10k-1} groups of k, we get that the first sage will make just k^{10k-1} correct guesses. The same holds true for every other sage, thus in total there are $10k \cdot k^{10k-1}$ correct guesses, which is not enough to have 11 correct guesses in each of k^{10k} situations.

1.4. [3, 4.2] Consider a sage. If the color of his hat coincides with the color of the hidden one, then he sees $2k$ hats of one color and $2k-2$ hats of another, thus he is sure that his hat is of minority color.

If his hat and the hidden one have different colors, then call this sage in this situation *a doubting sage*. Arguing analogously to 1.3 we prove that each sage makes a correct guess in exactly half of situations, in which he is a doubting sage. Indeed, let some sage i is doubting in some situation A. Construct the situation $h_i(A)$: take sage's hat and the hidden one and change there places. The sage i is still doubting, but since we did not change hats of all other sages, he must name the same color in both situations. Thus no strategy can guarantee more then $2k-1+\frac{2k}{2}=3k-1$ correct guesses.

So, we need to construct a strategy, where exactly half of doubting sages guess correctly in each situation. We do it in the following way.

Make the list of all $\binom{4k}{2k}$ situations and mark all doubting sages in each of them. We will take a pair of sage i and situation A_1, such that sage i is doubting in situation A_1, and thus also in situation $h_i(A_1)$. Set our strategy to order the sage i name the color of hat in the situation A_1 whenever he sees what he should see in the situation A_1. Thus he will make the right guess in A_1 and the wrong one in $h_i(A_1)$. Call $h_i(A_1)=A_2$ find another doubting sage in A_2 and do the same. Thus in A_2 there will be two sages, who's actions are already determined, and one of them makes the right guess, another one wrong. We continue this process until $A_k=A_1$. At this moment for each situation there are equal amounts of doubting sages, making right and wrong guesses. If not all the doubting sages have their actions determined — continue this process.

1.5. This problem was taken from [7]. We present you the solution after M. Ivanov, which in fact describes the same strategy as in [7], but is more elegant due to its algebraic formulation.

Let us label colors with residues 0, 1, 2 modulo 3. We need to find functions $f_A(D,B)$, $f_B(A,C)$, $F_C(B,D)$, $f_D(C,A)$ such that for any values of A, B, C, D at least one function coincides with the value of the corresponding variable modulo 3.

Let us try to find linear functions satisfying these conditions.

First, find the expressions of the form $A\pm B\pm C+\text{const}$, $A\pm C\pm D+\text{const}$, $A\pm B\pm D+\text{const}$, $B\pm C\pm D+\text{const}$ such that for any A, B, C, D at least one of these expressions is divisible by 3. For this, note that

$$(A+B+C)^2+(A-C+D)^2+(A-B-D)^2+(B-C-D)^2= \\ =3(A^2+B^2+C^2+D^2)\equiv 0 \pmod 3. \tag{1}$$

If for some A, B, C, D every expression

$$A+B+C,\quad A-C+D,\quad A-B-D,\quad B-C-D \tag{2}$$

is nonzero modulo 3, then the squares of these expressions have residues 1 modulo 3, and the sum (1) is not divisible by 3. It means that for any integers A, B, C, D, at least one of the expressions (2) is divisible by 3.

Now let $f_B = -A-C$, $f_D = C-A$, $f_A = B+D$, $f_C = B-D$. We can formulate a “recipe” for every sage: sage A says $B+D$, sage B says $-A-C$, sage C says $B-D$, sage D says $C-A$.

Remark. Formula (1) is just the product $(A^2+B^2+C^2+D^2)(1^2+1^2+1^2+0^2)$, rewritten with the help of the Euler formula

$$\begin{aligned}(A^2+B^2+C^2+D^2)(a^2+b^2+c^2+d^2) = \\ = (Aa+Bb+Cc+Dd)^2+(Ac-Ca+Db-Bd)^2+ \\ +(Ab-Ba+Cd-Dc)^2+(Ad-Da+Bc-Cb)^2.\end{aligned}$$

1.6. [2, p. 160]

2.1. Suppose that sages have a strategy which wins. Let v be the center of the foot, and let u_1, u_2, u_3 be terminal vertices. Temporarily let v be of the 1st color. Suppose that sages u_1, u_2, u_3 say colors h_1, h_2, h_3 according to their strategies.

Now perform another test: let now v be of the 2nd color. Suppose that sages u_1, u_2, u_3 say colors e_1, e_2, e_3 according to their strategies.

Now perform the final test. For every $i = 1, 2, 3$ we denote by d_i which was not said by the sage u_i in the first two test (if two colors are possible, we choose any one). For every i, we assign the color d_i to the vertex u_i. Since now v knows the colors of all his neighbors, we can predict his answer with respect to his strategy. One of the colors 1 and 2 does not coincide with this answer, so we assign v this color, and sages loose.

2.2. This is Lemma 8 from [1].

Using induction on the number of vertices, we prove the following statement. Let T be any tree, let v be any its vertex, and let c_1, c_2 be two arbitrary colors. Suppose that sages have already chosen a strategy Γ. Then there exists a distribution of hats into vertices, such that sages loose and, moreover, the vertex v has either color c_1 or color c_2.

Base of induction: if T has only one vertex. This is trivial.

Now we prove the induction step. If we delete the vertex v, the tree T will split into parts $T_1, T_2, \dots$. Let us denote by u_1, u_2, …the vertices in these subgraphs, which were adjacent to v in T. As we did in the solution of the previous problem, we perform two tests. In the first one, we color vertex v in the color c_1 and for every i consider all the distributions of hats in the subtree T_i which are losing for sages if sages use strategy Γ in T_i. Let H_i be the set of colors which u_i can take in these losing distributions. In the second test, let v be of color c_2 and for every i let E_i be the set of colors which u_i can have in all the losing distributions.

Note that in both experiments, the strategies of sages on every tree T_i differ only by the functions of the sage u_i. It means that if we manage to fix the hat color in the vertex u_i, then for every distribution of hats on the tree T_i the other sages would say the same color in both experiments.

By the induction hypothesis, each set H_i and U_i contains at least two elements, hence for every i the intersection of H_i and E_i is nonempty. For each i, choose any color d_i from $H_i \cap E_i$. Now we can construct the losing distribution: each u_i will be of color d_i, every tree T_i will be colored in such a way that the sages loose, and it remains to color v. Since we know the colors of all its neighbours, we know the answer of v. It does not coincide with one of c_1 and c_2, so we assign v this color. Now the sages loose.

2.3. [7, Lemma 2c]

Suppose that if $\ell_+(s_{m-1}s_m) \geqslant 2$, then this chain can be extended to the right by adding a vertex s_{m+1} in such a way that $s_1s_2\dots s_{m+1}$ is again a disproving chain for which the inequality $\ell_+(s_ms_{m+1}) \geqslant 2$ is held.

Indeed, potentially we have two (or even three) such extension, denote them by s_mv_1 and s_mv_2. Consider the short disproving chains s_mv_1w и s_mv_2w, and let us call it *perspective* if $v_i \neq f_{m+1}(s_m, w)$ (we take any of them if both satisfy this condition). In the same way we choose perspective chains for two other values of w. Now we have three perspective chains, and at least two of them have the same color (either v_1 or v_2) of the next vertex. So let s_{m+1} be this value of v_i.

Now without loss of generality let us assume that $\ell_-(s_1s_2) = 3$, $\ell_+(s_{m-1}s_m) \geqslant 2$. We can unlimitedly extend it to the right. It remains to check that we can loop it. Just before this, we have a long disproving chain $xs_1s_2\dots s_{n-1}y$, where x has three possibilities and y has at least two possibilities. So there exists $x = y$ such that $x \neq f_n(s_{n-1}s_1)$.

We obtained a cyclic disproving chain, and the sages loose.

2.4. [7, Lemma 2d]

If there exists a two-element disproving chain s_1s_2 such that $\ell_-(s_1s_2) + \ell_+(s_1s_2) > 4$, then the sages loose by the previous problem.

On the other hand, note that for a fixed s, we have

$$\ell_+(ss_1) + \ell_+(ss_2) + \ell_+(ss_3) = 6 \tag{3}$$

(here s_1, s_2, s_3 are three distinct colors). Indeed, if we fix color w, then there exists exactly two colors s_i such that $s_i \neq f(s, w)$. Since there are three possibilities for w, there are six possible continuations.

It means that

$$\sum_{s_1,s_2} \ell_+(s_1s_2) = 18. \tag{4}$$

Hence, the mean value of $\ell_+(s_1s_2)$ is 2. We can similarly show that the mean value of $\ell_-(s_1s_2)$ is 2.

Now let us solve the problem. Note that if for any two-element chain s_1s_2 the inequality $\ell_-(s_1s_2) + \ell_+(s_1s_2) < 4$ is held, then there exists another chain $s_1's_2'$ such that $\ell_-(s_1's_2') + \ell_+(s_1's_2') > 4$, and the sages loose.

Hence, the winning strategy can exist only if $\ell_-(s_1s_2) + \ell_+(s_1s_2) = 4$ for any s_1, s_2.

2.5. [7]

Construct a strategy of sages on the cycle of $N = 3n$ vertices, such that the emcee could not construct a disproving chain.

First, we deduce from Problems 2.12 and 2.13 that for a winning strategy, the number of right pairs of colors ab, where $a \in V_i$, $b \in V_{i+1}$, is the same for all i. (In the same way the number of left pairs is equal and the number of inert pairs is equal). It would imply that any chain, which disproves a winning strategy, contains links of the same type (i.e., either all the pairs of colors are right, or all the pairs of colors are left, or all the pairs of colors are inert), in this case all the chain will be called right/left/inert. Indeed, as we see in the solution of Problem 2.13, any left pair of colors ab_1 has a unique extension to the left to a more long chain c_1ab_1, and the pair c_1b is again left. In the same way a right chain can be uniquely extended to the right in such a way that its right link will be again a short right chain. Hence, if there exists a chain disproving all the sages, it contains links of the same type.

Now we find a strategy for the sages such that for all i, there are three right pairs ab, $a \in V_i$, $b \in V_{i+1}$, three left pairs and three inert pairs. So elements of V_i can be enumerated in such a way $V_i = \{v_1^i, v_2^i, v_3^i\}$ that the chains $v_1^1 v_1^2 v_1^3 \ldots$, $v_2^1 v_2^2 v_2^3 \ldots$, $v_3^1 v_3^2 v_3^3 \ldots$ are right. We give here the beginnings of these chains, but it can occur that if we loop them, they do not loop with period N, and for the extension $v_1^1 v_1^2 v_1^3 \ldots v_1^N v_1^{N+1}$ it turns out that $v_1^{N+1} = v_2^1$ or $v_1^{N+1} = v_3^1$. Denote $v_1^{N+1} = v_{\sigma(1)}^1$, $v_2^{N+1} = v_{\sigma(2)}^1$, $v_3^{N+1} = v_{\sigma(3)}^1$, clearly, σ is a permutation of the three-element set. So the aim of the sages is the following: invent a strategy such that local disproving chains could not loop in an N-element chain because of a permutation σ which has no fixed elements. The same should be true for left chains and for inert chains. Let us examine these chains using the enumerations of colors introduced above.

We have three left pairs ab, $a \in V_i$, $b \in V_{i+1}$, we may assume that these are pairs $v_1^i v_3^{i+1}$, $v_2^i v_1^{i+1}$, $v_3^i v_2^{i+1}$. There are also three left pairs among ab, $a \in V_{i-1}$, $b \in V_i$. Notice that a pair $v_3^{i-1} v_3^i$ is right, hence, the chain $v_3^{i-1} v_3^i v_1^{i+1}$ is not a short disproving chain, consequently, $v_3^{i-1} v_2^i v_1^{i+1}$ is a disproving chain, and this means that the pair $v_3^{i-1} v_2^i$ is a "disproving extension" to the left of the pair $v_2^i v_1^{i+1}$, which means that the pair $v_3^{i-1} v_2^i$ is again left. Using the same reasoning for the other pairs of indices, we obtain that for all i ($i = 1, 2, \ldots, N$) the set of left pairs ab, $a \in V_i$, $b \in V_{i+1}$, consists of pairs

$$v_1^i v_3^{i+1}, \qquad v_2^i v_1^{i+1}, \qquad v_3^i v_2^{i+1}.$$

But then the left chain which begins with the color v_1^1 has the form $v_1^1 v_3^2 v_2^3 v_1^4 \ldots$, so its $(N+1)$th element (recall that $3 \mid N$) has the form $v_{\sigma(1)}^{N+1}$. Hence, the left chain will not loop if the permutation σ has no fixed elements. The same is true for the inert chains.

It remains to describe the strategy which will provide such a picture. Suppose that all the sages use the same strategy:

$$f_i = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 1 \end{pmatrix}, \qquad i = 1, 2, \ldots, N$$

where the element in the pth row and in the qth column is equal to $f_i(p, q)$, and at the end $v_i^{N+1} = v_{\sigma(i)}^1$, where $\sigma : 1 \to 2 \to 3 \to 1$ is an appropriate permutation of the three-element set. In other words, if $v_1^i = 1$, $v_2^i = 2$, $v_3^i = 3$ for $1 \leqslant i \leqslant N$, then $v_1^{N+1} = 2$, $v_2^{N+1} = 3$, $v_3^{N+1} = 1$. This is due to the property $f_i(\sigma(p), \sigma(q)) = \sigma(f_i(p, q))$, which can be easily checked.

We leave to the reader the proof of the fact that this strategy has equal number of right, left and inert chains.

2.6. [7]

2.7. [1, Theorem 7]. The statement of Problem 3.4 hints that one or the parts of the graph must contain at least $k - 1$ vertices. It happens that this estimation is exact.

Let G be a complete bipartite graph with $n = k - 1$ vertices on the left side and $m = k^{k^n}$ vertices on the right side. Let C denote the set of all k-colorings of the left side of G. Note that $|C| = k^n$ and $m = k^{|C|}$, hence m is equal to the number of mappings from C to $\{1, 2, ..., k\}$. Pick a one-to-one correspondence between the vertices on the right side of G and the mappings from C to $\{1, 2, ..., k\}$, and let each vertex on the right side of G guess its color using the corresponding mapping.

We need the following lemma.

Lemma. Let c_R denote a fixed coloring of the right side of G, and let C' denote the set of all colorings c_L of the left side of G such that the combined coloring (c_L, c_R) causes every vertex on the right side to guess its color incorrectly. Then $|C'| < k$.

Now it's time to define the guessing strategies used by the vertices on the left side of G. Given the coloring of the right side, the set C' defined in the lemma above has at most $n = k - 1$ elements. So let

$c_1, c_2, \dots, c_n$ be a list of colorings which contains every element of C'. For $i = 1, 2, \dots, n$, vertex i on the left guesses that its color is $c_i(i)$. This guessing strategy (combined with the guessing strategy for the vertices on the right side as defined above) guarantees at least one correct answer. This is because the above lemma guarantees that at least one vertex on the right side guesses correctly unless the coloring of the left side belongs to C'. But if the coloring of the left side belongs to C', then it is equal to c_i for some $i \in \{1, 2, \dots, n\}$, in which case vertex i on the left guesses its color correctly.

It remains to prove the lemma. The proof follows from noting that if C' contains k distinct elements $c_1, c_2, \dots, c_k$, then there exists a function f from C to $\{1, 2, \dots, k\}$ which assumes k distinct values on the set $\{c_1, \dots, c_k\}$. Let v denote the vertex on the right side of G corresponding to f. Since the set $\{f(c_1), f(c_2), \dots, f(c_k)\}$ contains all k colors, we must have $f(c_i) = cR(v)$ for some i in 1, 2,..., k. Thus, the combined coloring (c_i, c_R) causes vertex v to guess its color correctly, contradicting our assumption that c_i belongs to C', ending the proof.

2.8. This is Lemma 1 from [4]. We re-write this proof here in a more readable way.

The sages win on the graph G, when they have hats of q colors. Let us call these colors *warm*. The sages win on the graph K_r, when they have hats of r colors. Let us call these colors *cold*. In order to color the graph $\widetilde{G}$ into qr colors, we have to assign a warm color and a cold color to each vertex. During the test, the sages will also say two colors: a warm one and a cold one.

To choose a cold color, the sages will look only on the other sages in their copy of K_r (and taking into account only the cold components of their colors). Then for every copy of K_r, exactly one sage will guess his cold color correctly, we call them *lucky*. Every sage can understand which sage is lucky in every adjacent copy of K_r. To guess his warm color, every sage uses his strategy on the graph G, assuming that his neighbors on G have the colors of lucky sages on the K_rs corresponding to vertices of G and taking into account only the warm component of their color. Then at least one lucky sage will guess his color correctly.

2.9. This graph can be obtained from Problems 1.5 and 2.8.

2.10. Let us perform the same computation as we did in Problem 1.3. Let us sum up the number of all guesses in all the distributions of hats. On one hand, there are $3 \cdot 3^2$ guesses. On the other hand, there are 3^3 distributions of hats, so if the strategy of sages is winning, there is exactly one guess in every distribution.

Now fix any strategy and present a distribution of hats where at least two sages guess. Let us assign any color to the sage C. Then give A the hat of color which he says with respect to his strategy for this color of C. произвольную шляпу. Now let B be of the color which he says with respect to his strategy for these colors of A and C.

2.11. Denote the sages by A, B, C, D, and suppose that A does not see B. First of all we show, that there exist two three-element chains $a_1d_1c_1$ and $a_1d_1c_2$ such that A and D do not guess. For this, as we did in the proof of Problem 2.4, consider all the 6 distributions of colors to A and D in such a way that A does not guess (his strategy depends only on the color of D), and 18 possibilities to extend this chain in the direction of C. Since in total D guesses only in 9 situations, there is a color d_1 which he says at most three times. Then among the chains with the beginnings a_1d_1 or a_2d_1 there are at least three losing for D, and by the Pigeonhole Principle there are two three-element chains $a_1d_1c_1$ and $a_1d_1c_2$ such that A and D loose. Give A and D hats of colors a_1 and d_1, respectively.

Now consider the strategy of B. Let $f_B(a_1, c_1) = b_1$, $f_B(a_1, c_2) = b_2$. Let us give him a hat of the third color b_3 (of any possible color, if $b_1 = b_2$).

Notice that now we know the colors of C's neighbors: B has color b_1 and D has color d_1. But $f_C(b_1, d_1)$ does not coincide with one of c_1, c_2. If we give the sage C the non appropriate color, the sages loose.

2.12. [7, Lemma 3a] We use formula (4) and the analogous formula for ℓ_-:

$$\sum_{a,b} \ell_+(ab) = \sum_{a,b} \ell_-(ab) = 18.$$

Since we consider a winning strategy, for any a, b $\ell_+(ab) + \ell_-(ab) = 4$. Hence, any summand 1, 2, 3 in the first sum corresponds to a summand 3, 2, 1 in the second sum. It means that both sums contain the same number of 1s and 3s.

2.13. [7, Lemma 3d] Let $V_{i+1} = \{b, b_1, b_2\}$. Then, as in Formula (3), and taking into account that $\ell_-(s_1 s_2) + \ell_+(s_1 s_2) = 4$ we have

$$\ell_-(ab) + \ell_-(ab_1) + \ell_-(ab_3) = 6.$$

Since $\ell_-(ab) = 3$, two other summands are 1 and 2, we may assume that $\ell_-(ab_1) = 1$, it means that there exists a short disproving chain, say, $c_1 a b_1$. But then $\ell_-(c_1 a) = 1$, since in the opposite case $\ell_-(c_1 a) + \ell_+(ab_1) \geqslant 5$ and the strategy is losing by Problem 2.3. We apply the analogous formula

$$\ell_-(c_1 a) + \ell_-(c_2 a) + \ell_-(c_3 a) = 6$$

and see that $\ell_-(c_1 a) = 1$, hence, two other summands are 2 and 3. We are done.

2.14. [1, Theorem 16.iii]

2.15. [4]

3.1. All the sages, besides the last one, say the color opposite to the color they see. The last sage says the color he sees.

3.2. [1, Example 6] It follows from Problem 3.1 that at least c sages can guess the color of their hat. In Problem 3.5 you can find the example of the graph for which the number of sages is greater than the number of independent cycles.

3.3. [1, Lemma 4] Suppose that the graph becomes acyclic after deleting vertices v_1, v_2, …, v_a. Number the remaining vertices v_{a+1}, …, v_n in such a way that the numbers along every edge decrease. In other words, for the last $n - a$ vertices all the edges are directed to the left. Now let us arbitrarily distribute hats among the first a sages. For every next sage, the colors of hats which he sees are already determined, so his answer with respect to his strategy is known. We can give him a hat of another color so that he does not guess.

For this distribution of hats only the first a sages can guess their color.

3.4. This is Theorem 5 from [4]

Take any sages' strategy f and prove that it is losing.

Let A be the set of all but one colors of hats, $|A| = k - 1$. If a is a color, we denote by w_a the collection of $k-2$ colors $(a, a, \dots, a)$. The sages from L will always get the same hats of colors w_a, where $a \in A$.

Let us enumerate the vertices of R by r_1, r_2, … r_s in such a way that the numbers along every edge decrease. To simplify the reasoning, we consider only the case when every sage from R sees all the sages with less numbers. Construct the collection of colors $Y = \{y_1, \dots, y_s\}$ for the sages in R. For this, we consequently take

$$\begin{aligned}
&y_1 \notin \{f_{r_1}(w_a),\ a \in A\};\\
&y_2 \notin \{f_{r_2}(w_a, y_1),\ a \in A\};\\
&\quad \dots\\
&y_s \notin \{f_{r_s}(w_a, y_1, y_2 \dots, y_{s-1}),\ a \in A\}.
\end{aligned}$$

Let us explain these formulas for y_s. The sage in the vertex r_s sees all the sages from the part L (their colors are given by the collection w_a), and he sees the sages from part R with smaller numbers. Hence, we know his answer $f_{r_s}(w_a, y_1, y_2 \dots, y_{s-1})$ according to his strategy. The color a takes values in the $(k-1)$-element set A; the set in the right hand of the formula for y_s contains at most $k-1$ elements, hence, we can choose the appropriate color y_s according to this formula.

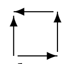
Now the set of colors Y is constructed. Let ℓ_1, ℓ_2, $\dots \ell_{k-2}$ be the vertices of L. Choose a color $b \in A$ which coincides with none of the colors $f_{\ell_1}(Y)$, $\dots$, $f_{\ell_{k-2}}(Y)$. Then no sage guesses his color for the distribution (w_b, Y).

3.5. [1, Example 4]

4.1. Let us present an equality analogous to (1), and with its help construct linear functions, giving the strategies of sages. Since this equality is too long, we will not write it here, so we need some preliminary work.

By an N-dimensional hypercube Q_N we mean a graph, which contains 2^N vertices, enumerated by N-digit numbers in the binary number system, and the edges join numbers differing only in one binary digit. The next constructions can be applied to any hypercube, however they are applicable to the problem about hats only for $N \equiv 2 \pmod 3$.

Lemma. The edges of the hypercube Q_N can be oriented in such a way that every 4-cycle in Q_N will contain 3 edges pointing to one direction of bypass of this cycle and one edge in the opposite direction.

Proof. Induction on N. Base $N = 2$. 

The step of induction. Suppose that we have already oriented the graph Q_N. We may assume that Q_{n+1} consists of two copies of Q_N, the “left” one and the “right” one, and for every vertex of the left copy there is an edge to the corresponding vertex of the right copy. Suppose that we have already oriented all the edges in the left copy according to the induction hypothesis, and let us orient the right copy in the opposite way. For the edges between the copies, all the arrows will point to the right. It is easy to see that this orientation satisfies the conditions. □

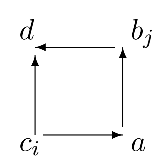
Now let us take independent variables, one for each vertex of Q_N.

Recall that every vertex of Q_N has degree N. Suppose that a is an arbitrary vertex of the graph; b_1, b_2, $\dots$ are vertices such that there are arrows from a to them, c_1, c_2, $\dots$ are vertices such that there are edges from them to a. For any a consider the expression f_a, equal to the square of the linear combination

$$f_a = (a + b_1 + b_2 + \dots - c_1 - c_2 - \dots)^2. \tag{5}$$

Consider the sum $\sum f_a$ of these squares within all the vertices of the graph. Now open all the brackets. For every vertex a, the summands of the form a^2 will appear in this sum with multiplicity $N + 1$, since every such summand appears from the brackets f_a, f_{b_1}, f_{b_2}, $\dots$, f_{c_1}, f_{c_2}, $\dots$ and only from them. Moreover, every edge ab corresponds to a summand $+2ab$, which appears from the bracket f_a, and a summand $-2ab$, which appears from the bracket f_b. While opening the other brackets, such summands can not appear, so they all vanish.

There is one more type of summands, from the bracket f_a we obtain summands of the form $-2b_j c_i$, let us examine them. Suppose that a has number 00, b_j has number 01, c_i has number 10. Consider also the vertex d with number 11 (we write here only the bytes where the numbers differ). Clearly, $f_d = (d - b_j - c_i + \dots)^2$, so the summand $2b_j c_i$ appears from f_d with the sign “plus”. It will vanish. Another possible orientations of the edges of the cycle can be treated in the same way.



So $\sum_a f_a = (N+1) \cdot \sum_a a^2$.

Now we come back to our sages. Let $N \equiv 2 \pmod 3$. Then the sum $\sum_a f_a$ is divisible by 3. It consists of 2^N summands. Clearly, either $f_a \equiv 0$ or $f_a \equiv 1 \pmod 3$. Hence, at least one of the summands f_a must be zero modulo 3 (for odd N there are at least two such summands). Using the notation of (5), we demand that for every vertex a the sage in this vertex uses the hypothesis $c_1 + c_2 + \ldots - b_1 - b_2 - \ldots$ Then the sage sitting in a vertex a such that $f_a \equiv 0 \pmod 3$ will guess the color of his hat.

4.2. [1, Lemma 11] To describe the strategies, we will use the hypercube (see the text before the formulation of the problem).

Let is cut the hypercube into layers: the ith layer will be formed by all the vertices with the sum of coordinates equal to i. The number of non-oriented edges, going from a vertex v to the vertices of the next layer, will be called the *upper degree* $\operatorname{udeg} v$ of this vertex, and the number of edges going to the vertices of the previous layer will be called the *lower degree* $\operatorname{ddeg} v$ of the vertex.

Consider the edge between the ith and $(i+1)$th layers and the corresponding sage (=the coordinate which changes), it equals 1 in the $(i+1)$th layer and 0 in the ith layer. The strategy gives the orientation of this edge: if it points from the ith layer to the $(i+1)$st layer, then the sage will guess when his hat is of color 1 and will loose in the opposite case. If the edge points from the $(i+1)$th layer to the ith, then the sage guesses when his hat is of color 0 and looses in the opposite case. If for every vertex v of the ith layer the number of edges pointing from the $(i+1)$th layer to this vertex equals $[\operatorname{udeg} v/2]$, and the number of edges pointing from the $(i-1)$st layer to this vertex equals $[\operatorname{ddeg} v/2]$, then it is easy to see that we get a balanced strategy.

Construct a balanced strategy, i. e., orient the edges in order to fulfill the properties of upper and lower degrees mentioned above. The idea is clear: we take any edge between the ith and $(i+1)$st layers, put any orientation on it and construct an oriented path, adding new edges in such a way that the path remains between the ith and $(i+1)$st layers. If it is not possible to extend this path (in both directions) but not all the edges are oriented, we start constructing a new path, etc.. When we orient all the edges, we obtain a balanced strategy.

4.3. [1, Proposition 13] If the number of sages is even, the optimal strategy is a strategy such that for every vertex of the hypercube, the incoming degreee and the outcoming degrees are equal. In this case, one can construct an oriented Euler path. Now the strategy of the ith sage is the orientation of edges, parallel to the ith coordinate line. Note that one half of the vertices of the hypercube in in the (left) face $x_i = 0$, and the other half in in the right face $x_i = 1$. The arrows pointing to the left correspond to the case when the sage says 0 and the arrows pointing to the right correspond to the case when the sage says 1. The Euler path contains an equal number of such arrows.

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