Point-Line Incidences.

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Figure 1: Paul Erdos

Once upon a time there was a mathematician named Erdos, who invented a lot of striking problems. One day he gave us the following riddle.

Given a finite set $\mathcal{L}$ of lines and a finite set P of points, let $I(\mathcal{L}, P)$ denote the number of incidences between $\mathcal{L}$ and P, i.e., the number of pairs (l, p), $l \in \mathcal{L}$, $p \in P$, such that $p \in l$.

By $I(n,m)$ denote the largest $I(\mathcal{L}, P)$ among all pairs $(\mathcal{L}, P)$ provided that $|\mathcal{L}| = n$, $|P| = m$. (Here and below the symbol $|A|$ denotes the cardinality of the set A.)

Main question: How to estimate $I(n, m)$?

P. Erdos conjectured: there exists a constant C such that

$$I(n, n) \leqslant C\left(n^{4/3}\right).$$

Notice that this bound is evidently better than the trivial one $I(n, n) \leqslant n^2$.

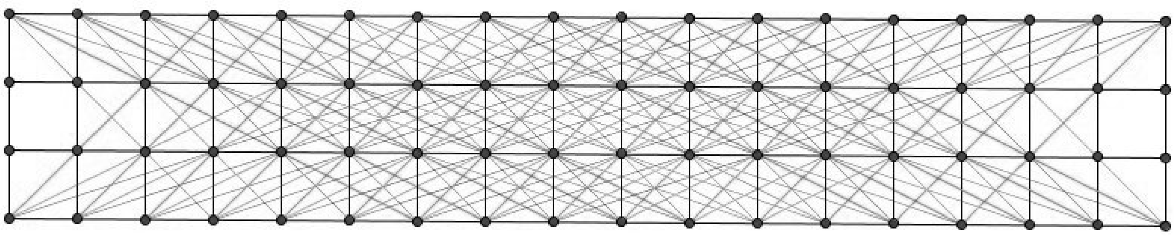


Figure 2: Many point-line incidences

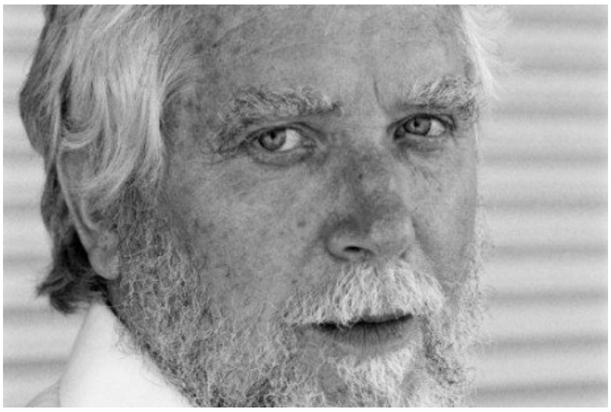


Figure 3: Endre Szemeredi

Figure 4: William Trotter

In 1983, Szemeredi and Trotter managed to prove the Erdos conjecture. This result is called the **Szemeredi–Trotter theorem**.

Our main aim is to prove it using two different approaches. Along the way, we will discover some useful tricks and handle a lot of geometric and combinatorial problems.

1 Combinatorics

1.1 Introduction. Some Combinatorial Problems in Geometry.

1.1. Consider a finite set A of points in the plane. Prove that there exists a line, which divides the plane into two half-planes such that each open half-plane contains at most $\lfloor |A|/2 \rfloor$ points of A.

Proof. Let $\{a_1, a_2, \dots a_n\}$ be the given set and let L be an arbitrary half-plane, bound of which is parallel to no segment joining any two points of the set. We define L_i as a half-plane obtained by parallel shift of L, and bound of which passes through a_i. We note that for any j, $0 \leqslant j \leqslant n$, there exists a half-plane L_{i_j} containing exactly j points of the given set $\{a_1, a_2, \dots a_n\}$. This implies that the bound of $L_{i_{[n/2]}}$ is a required line. □

1.2. Consider $2n$ points in the plane. Prove that the points can be divided into n pairs in such a way that there is no intersection between line segments joining points in pairs.

Proof. We will prove it by contradiction. Assume that points could not be divided in the required way. We note that $2n$ points can be paired in only finitely many ways. This impies that there exists a partition with the minimal sum $\sum'$ of all segment lengths. By assumption, there exist two intersecting segments AB and CD. Then the sum of AC and BD is less than the sum of AB and CD. This contradicts the minimality of the sum $\sum'$. □

1.3. Consider n points in the plane so that no three points lie on a line. Construct a non self-intersecting polygonal line joining the given points.

Proof. Consider a polygonal line Γ joining the given points, which has the minimal length. Let us prove by contradiction that the given polygonal line is required. Let two segments AC and BD intersect each other. From the triangle inequality it follows that the sum of AB and CD, also as well as the sum of AD и BC, is less than the sum of AC and BD. Let Ω be a union of segments of Γ, except AC and BD. Consider $\Gamma_1 = \Omega \cup \{AB, CD\}$. If Γ_1 is not connected, then there is no path in Ω joining A and B, also as well as no path joining C and D. Consider another union $\Gamma_2 = \Omega \cup \{AD, BC\}$. If Γ_2 is not connected, then there is no path in Ω joining A and D, also as well as no path joining C and B. This implies that if both collections Γ_1 and Γ_2 are not connected, then there is no path in Γ joining A and B. This contradicts the connectivity of Γ. Without loss of generality, let Γ_1 be connected. But the sum of lengths of segments in Γ_1 is less than in Γ. Here we come to a contradiction. $\square$

1.4. Consider n points and n pairwise non-parallel straight lines in the plane. Prove that points and lines can be enumerated from 1 to n in such a way that any two segments of perpendiculars from the corresponding points onto the lines don't intersect each other.

Proof. Consider an enumeration, at which the sum of length of altitudes is minimal. By F_i denote i-th point, and by l_i denote i-th line. Assume that, for some i anf j, segments of perpendicular dropped from F_i and F_j are intersected in E by each other. Let A and B be the feet of perpendiculars dropped from F_i and F_j onto l_i and l_j correspondently. Draw altitudes dropped from F_i and F_j onto l_j and l_i with foot C and D correspondently.

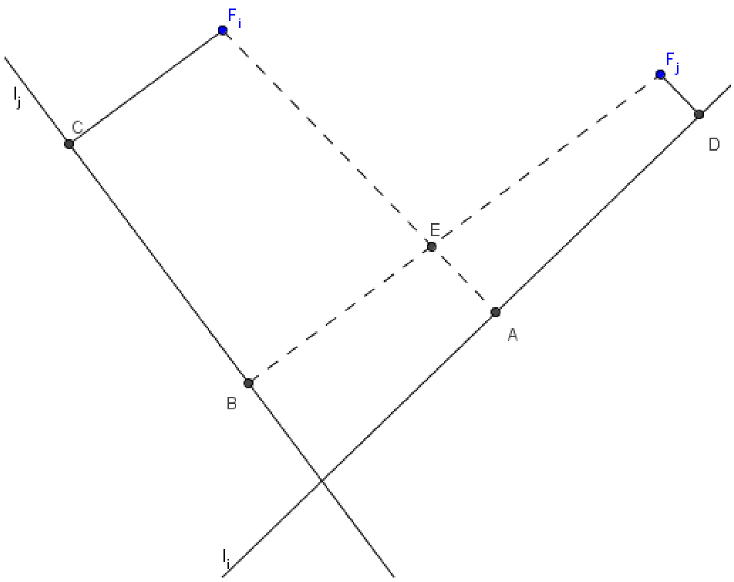


Figure 5:

Thus, we have

$$F_iA + F_jB = F_iE + EA + F_jE + BE = (EA + F_jE) + (F_iE + BE) \geqslant$$

$$\geqslant F_jA + F_iB \geqslant F_jD + F_iC.$$

This means that we come to a contradiction. $\square$

1.5. Consider a non-convex polygon $A_1A_2\dots A_n$. Suppose two non-adjacent vertices A_i and A_j satisfy the following property: the polygon lies completely on one closed half-plane bounded by the line A_iA_j; then we reflect either the polyline $(A_i \dots A_j)$ or $(A_j \dots A_i)$ through the center of the segment A_iA_j. Prove that applying a finite number of reflections we must come to a convex polygon.

Proof. Let γ_i, $i \in [n-1]$, be the smaller angle between A_iA_{i+1} and X-axis. By γ_n denote the smaller angle between A_nA_1 and OX-axis. Consider a collection of pairs ("segment length", "angle") $\Omega = \{(|A_1A_2|, \gamma_1), (|A_2A_3|, \gamma_2), \dots (|A_nA_1|, \gamma_n)\}$. After each reflection, Ω will be recalculated for a new polygon. Note that the difference between any two collections is in the order of pairs. This implies that we can get a finitely many polygons. We also note that the area of the next polygon is more than the area of the current polygon. It follows that we will come to a point when the current polygon is convex. $\square$

1.2 Incidence Problems

1.6. Let $\mathcal{M} = \{M_1, \ldots, M_s\}$, $M_i \subset \{1, 2, \ldots, n\}$, $|M_i| = 3$, be an arbitrary collection of subsets of n-set. Suppose $|M_i \cap M_j| \neq 1$; then find the largest possible s.

Proof. Consider two intersecting sets $A \in \mathcal{M}$ and $B \in \mathcal{M}$. Let $F = A \cap B$, $G = A \cup B$ and $A = \{a, b, c\}$. Without loss of generality, $B = \{b, c, d\}$, $F = \{b, c\}$. Now let us see which sets can intersect G. We distinguish two cases.

Case 1: No other set of M contains F, i.e., $\nexists C \in \mathcal{M} : C \supset F, C \notin \{A{,}B\}$.

Let $C \cap (A \cup B) \neq \varnothing$. We have $\{a, d\} \subset C$, since $C \not\supseteq F$. Thus, either $C = \{a, b, d\}$, or $C = \{a, c, d\}$.

Notice that four sets $A, B, \{a, b, d\}$ and $\{a, c, d\}$ could be elements of $\mathcal{M}$ at one time. Moreover, no other set is intersected by $A \cup B$. Such union of four sets is said to be a *cookie.*

Case 2: There exist exactly $k + 2 \geqslant 3$ sets of $\mathcal{M}$ containing F:

$$\{A, B, A_1, A_2, \ldots, A_k\}.$$

Assume that $A_i = \{a_i, b, c\}$, and there exists $D \in \mathcal{M}$, $D \not\supseteq F$ such that

$$D \cap (A \cup B \cup A_1 \cup A_2 \cup \cdots \cup A_k) \neq \varnothing.$$

Without loss of generality, $D \cap A \neq \varnothing$, $D \cap A = \{a, b\}$ and $D = \{a, b, x\}$. But since $D \cap B \neq \varnothing$, $D \cap A_1 \neq \varnothing$ and $x \neq c$, then we have that $x = d$ and $x = a_1$ at one time. It means we come to a contradiction, and there is no such set D.

Note that this collection $A, B, A_1, A_2, \ldots, A_k$ can be a subset of $\mathcal{M}$. Such collection of $k + 2$ sets will be called a *flower with* $k + 2$ *petals.*

Thus, n-set is divided into disjoint cookies and flowers.

We note that a cookie has exactly 4 elements, while a flower with k petals has $k + 2$ elements. It follows that we get $s \leqslant n - 2t$, where t is the number of flowers.

This implies that we have the maximal $s = 4 \cdot \lfloor \frac{n}{4} \rfloor$ for $n \equiv 0, 1, 2 \, (mod\, 4)$, which is attained when n-set is divided into $\lfloor \frac{n}{4} \rfloor$ cookies (there may remain 1 or 2 elements not included in any set of $\mathcal{M}$). For $n \equiv 3 \, (mod\, 4)$, we have $s = n - 2$ if all elements are included in one flower. $\square$

1.7. Let $\mathcal{M} = \{M_1, \ldots, M_s\}$, $M_i \subset \{1, 2, \ldots, n\}, |M_i| = 4$, be an arbitrary collection of subsets of n-set such that $|M_i \cap M_j| \neq 2$. Denote by $S(n)$

the largest possible s. Prove that $\lfloor n/4 \rfloor^2 \leqslant S(n) \leqslant n(n-1)/4$.

Proof. 1) An upper bound. Consider an arbitrary element A of n-set. Let $\mathcal{F}_\mathcal{A}$ be a collection of elements of $\mathcal{M}$ containing A. Consider a collection $\mathcal{F}'_\mathcal{A}$ obtained from $\mathcal{F}_\mathcal{A}$ by removing A from each set of $\mathcal{F}_\mathcal{A}$. Any two sets of $\mathcal{F}'_\mathcal{A}$ are either disjoint, or intersected in two elements. Thus, we can apply the result of problem 1.6. We get that $\mathcal{F}'_\mathcal{A}$ has at most $(n-1)$ sets. The number of sets in $\mathcal{F}_\mathcal{A}$ is equal to the number of sets in $\mathcal{F}'_\mathcal{A}$. Hence, for any $A \in [n]$, the collection $\mathcal{F}_\mathcal{A}$ has at most $(n-1)$ four-element sets. This implies $s \leqslant \frac{n(n-1)}{4}$.

2) A lower bound. Here we give an example. Enumerate elements from 1 to n. Consider the following sets $\{4k+1, 4k+2, 4k+3, 4t\}$, where $k = 0, 1, \ldots, \lfloor \frac{n}{4} \rfloor - 1$ and $t = 1, 2, \ldots, \lfloor \frac{n}{4} \rfloor$. We have exactly $\lfloor \frac{n}{4} \rfloor^2$ such sets. It is easy to check that the considered collection is desired. □

Definition. Let $\mathcal{L} = \{l_1, \ldots, l_n\}$, $\ l_i \in P$, be an arbitrary collection of subsets of m-set $P = \{p_1, \ldots p_m\}$. A pair (l_i, p_k) is said to be an *incidence* if $p_k \in l_i$. Denote by $I(\mathcal{L}, P)$ the number of incidences formed by the elements of $\mathcal{L}$ and P. Define $I(n,m) = \max\limits_{|\mathcal{L}|=n,\, |P|=m} I(\mathcal{L}, P)$.

Our goal is "how to estimate $I(n,m)$ when P and $\mathcal{L}$ satisfy certain properties". Otherwise, we have only a trivial bound $I(\mathcal{L}, P) \leqslant mn$.

A typical interpretation of incidences is as follows:

Consider a blank matrix X with n rows and m columns. Rows correspond to elements of the set $\{l_1, \ldots, l_n\}$, and columns correspond to elements of the set $\{p_1, \ldots p_m\}$. We will put a star in the entry $x_{i,j}$ iff $p_j \in l_i$. Now we can reformulate the problem: "how many stars are there in the matrix?"

Use this interpretation if it is convenient. (see problem **1.11.**)

Further, it is supposed that $n, m, r \in \mathbb{N}$, and $\mathcal{L} = \{l_1, \ldots, l_n\}$ is an arbitrary collection of subsets of m-set $P = \{p_1, \ldots p_m\}$ such that $|l_i \cap l_j| \leqslant r$ for any $i \neq j$.

1.8. Let $r = 1$. Prove that

(a) $I(\mathcal{L}, P) \leqslant n^2 + m$, $I(\mathcal{L}, P) \leqslant m^2 + n$.

(b) $I(\mathcal{L}, P) \leqslant \sqrt{m(n^2 - n)} + m$, $I(n, m) \leqslant \sqrt{n(m^2 - m)} + n$.

Proof. (a) Let us prove $I(\mathcal{L}, P) \leqslant m^2 + n$. Divide subsets in $\mathcal{L}$ into two groups: subsets in $\mathcal{L}_1$ are incidence to at most one element of P, and let $\mathcal{L}_2 = \mathcal{L} \backslash \mathcal{L}_1$. Then $I(\mathcal{L}_1, P) \leqslant |\mathcal{L}_1| \leqslant n$. It remains to bound $I(\mathcal{L}_2, P)$.

Notice that any element p of P may have at most $m-1$ incidences with the elements of $\mathcal{L}_2$, since otherwise there exist two elements in $\mathcal{L}_2$ that have at least two common elements of P (p and another one). Thus, $I(\mathcal{L}_2,P) \leqslant m(m-1) \leqslant m^2$.

The other inequality follows using the geometric duality leading, in particular, to $I(n,m) = I(m,n)$.

(b) Use the above-mentioned analogue with a matrix. By r_i, $1 \leqslant i \leqslant n$, denote the number of stars in the corresponding row. Let $r_i \geqslant r_j$ for $i \leqslant j$ and let k be a such number that $r_k \geqslant 2$ and $r_i = 1$ for $i > k$. Then notice

$$\binom{m}{2} \geqslant \sum_{i=1}^{k} \binom{r_i}{2},$$

since otherwise there exist two rows which have 2 stars in the intersection of the same columns. Then we have

$$2n \cdot \binom{m}{2} \geqslant 2n \cdot \left(\sum_{i=1}^{k} \binom{r_i}{2} + \frac{n-k}{2} - \frac{n-k}{2} \right) \geqslant I^2 - nI \geqslant (I-n)^2.$$

□

1.9. Prove that

$$I(\mathcal{L}, P) \leqslant \sqrt{mr(n^2-n)} + m$$

Proof. Use the same arguments as in the proof of problem **1.8(b)** (the factor r appears on the left-hand side of the considered chains of inequalities). □

1.10. Let $r = 1$.

(a) Find $\max I(\mathcal{L}, P)$ if $n \leqslant 3$.

(b) Find $\max I(\mathcal{L}, P)$ if $m \geqslant C_n^2$. Find a configuration such for which the maximal bound is obtained.

Добавить решения.

Proof. (a) Assume $|\mathcal{L}| = n = 1$, then $\max I(\mathcal{L}, P) = m$. For $|\mathcal{L}| = n = 2$, we get $\max I(\mathcal{L}, P) = m+1$, since otherwise (if $I(\mathcal{L}, P) \geqslant m+2$) there exist at least two common elements of P contained in l_1 and l_2. If $|\mathcal{L}| = n = 3$, then we have $\max I(\mathcal{L}, P) = m + 3$,

(b) $\max I(\mathcal{L}, P) = m + \binom{n}{2}$. The maximum is attained when all n lines are in general position. □

1.11. Let an 13×13 square be divided into the unit squares. Some centers of unit squares are marked in such a way that there is no rectangle with vertices being in marked points, sides of which are parallel to the sides of the square. Find the largest possible number of marked points.
Answer: 52 points could be (see Figure 6). An upper bound on the number of stars follows from the problem **1.12** (b).

	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{A}_4$	$\mathbf{A}_5$	$\mathbf{A}_6$	$\mathbf{A}_7$	$\mathbf{A}_8$	$\mathbf{A}_9$	$\mathbf{A}_{10}$	$\mathbf{A}_{11}$	$\mathbf{A}_{12}$	$\mathbf{A}_{13}$
l_1	*	*		*						*			
l_2		*	*		*						*		
l_3			*	*		*						*	
l_4				*	*		*						*
l_5	*				*	*		*					
l_6		*				*	*		*				
l_7			*				*	*		*			
l_8				*				*	*		*		
l_9					*				*	*		*	
l_{10}						*				*	*		*
l_{11}	*						*				*	*	
l_{12}		*						*				*	*
l_{13}	*		*						*				*

Figure 6:

1.12. 100 (a) little mouses are nibbling 1000 (b) pieces of cheese. Each mouse eats some pieces making holes. But any two mouses leave holes in at most 10 (c) common pieces of cheese.

(a) Prove that the number of holes is at most 11000 ($b + a\sqrt{bc}$).

(b) Prove that the number of holes is at most 10500 $\left(\frac{b+\sqrt{b^2+4bca(a-1)}}{2}\right)$.

Proof. Suppose that each piece of cheese is eaten. Let d_i be the number of pieces bitten by i mouses. By I denote the number of holes. Then we have

$$\sum_i d_i = b, \quad \sum i d_i = I, \quad \sum \frac{i(i-1)}{2} d_i \le c\frac{a(a-1)}{2}.$$

Applying the inequality of arithmetic and quadratic means, we obtain

$$I - b = \sum (i-1) d_i = b\frac{\sum (i-1) d_i}{\sum d_i} \le b\sqrt{\frac{\sum (i-1)^2 d_i}{\sum d_i}} \le \sqrt{b(ca(a-1) - (I-b))}.$$

Thus, $I \le b + a\sqrt{bc}$. In order to get a more precise bound, we should reduce the above-mentioned inequality to

$$(I-b)^2 + b(I-b) - bca(a-1) \le 0$$

and solve it. $\square$

1.3 Ham Sandwich Theorem.

1.13. Prove that for a finite set of points in general position in the plane each colored "red" or "blue", there is a line that simultaneously bisects the red points and bisects the blue points. ("Bisecting" means that each side contains less than half of the total number of points.)

Proof. By $R(\phi)$ denote the set of directed lines which make an angle ϕ with OX-axis clockwise and bisect red points. In the same way, by $B(\phi)$ denote the set of directing lines bisecting blue points having the slope ϕ. For any fixed ϕ, $R(\phi)$ is either a strip or a line. We let $r(\phi)$ be the "middle" directed line of $R(\phi)$, and $b(\phi)$ is denoted similarly. Let us prove that $r(\phi_0) = b(\phi_0)$ for a certain ϕ_0. If $r(0) = b(0)$, then $\phi_0 = 0$. Otherwise without loss of generality, we may assume that $R(0)$ is on the left of $b(0)$ if we are looking across the direction of $r(0)$. One will change ϕ from 0 to π. Notice that $r(\pi)$ is on the right of $b(\pi)$, since $r(\pi)$ and $b(\pi)$ look like $r(0)$ and $b(0)$, except the reverse directions. For $r(\phi)$ and $b(\phi)$ are continuous on $\phi \in [0,\pi]$, we finally obtain that there exist a required ϕ_0. □

1.14. Prove that for a finite set of points in general position in space each colored "red", "blue" or "green", there is a plane that simultaneously bisects the red points, the blue points and the green points.

Proof. We generalize the previous proof. Fix a spherical coordinate system in the space. This means the position of a point is specified by three numbers: the radial distance of that point from a fixed origin, its polar angle measured from a fixed zenith direction, and the azimuth angle of its orthogonal projection on a reference plane that passes through the origin and is orthogonal to the zenith, measured from a fixed reference direction on that plane (See Figure 7.)

By $R(\varphi, \psi)$ denote the set of planes, normal vectors of which are equal to $\vec{n} = (1, \varphi, \psi)$, and that also bisect red points. In the same way, by $G(\varphi, \psi)$ and by $F(\varphi, \psi)$ denote the sets of planes bisecting blue and green points correspondently. We choose the middle planes of $R(\varphi, \psi)$, $G(\varphi, \psi)$ and $F(\varphi, \psi)$ and designates them by $r(\varphi, \psi)$, $g(\varphi, \psi)$ and $f(\varphi, \psi)$. Without loss of generality, assume that if we are looking along the direction of $\vec{n}$, then $r(0, 0)$ is above $g(0, 0)$ and $f(0, 0)$, and $g(0, 0)$ is lying between $r(0, 0)$ and $f(0, 0)$. Introduce

$$A(0, 0) = (dist(r(0, 0); g(0, 0)), -dist(g(0, 0); f(0, 0))),$$

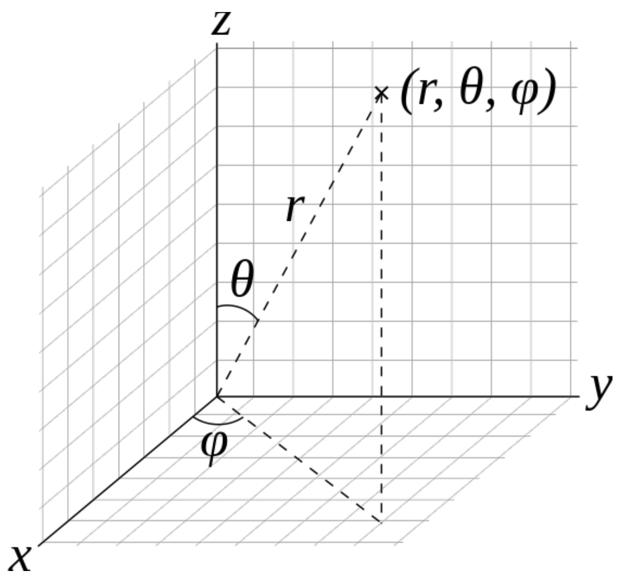


Figure 7: A spherical coordinate system

where $dist$ is a distance between planes. In the same way, define $A(\varphi, \psi)$ for all (φ, ψ) (a sign before distance depends on the positional relationship of the planes). Notice that the considered map is continuous on $\{0 \leqslant \varphi \leqslant 2\pi\} \times \{-\pi/2 \leqslant \psi \leqslant \pi/2\}$, and satisfy the following property $A(\varphi, \psi) = -A(\varphi + \pi, -\psi)$ for any $\varphi, 0 \leqslant \varphi \leqslant \pi$. There remains to prove that there is a pair (φ, ψ), at which the function A vanishes.

Suppose that $A(0, \pi/2) \neq (0, 0)$. Fix an arbitrary ψ. We have that the image $A(\varphi,\psi), \varphi \in [0,2\pi]$ is a closed curve (it may be a point). If $\psi = 0$, then from symmetry of A, it follows that such image is symmetrical with respect to (0,0), and the point (0,0) belongs to the interior bounded by this loop. Shifting a parameter ψ continually changes the interior of the loop. In particular, this means that there is a ψ_0 such that $A(\varphi,\psi_0) = (0,0)$ for some φ.

□

1.15. Consider $2m$ points in general position in the plane such that m points are "red" and the others are "blue". Prove that the points can be divided into n pairs in such a way that each pair consists of a red point and a blue point, and no two segments joining points in pairs intersect each other.

Proof. One can enumerate blue points and red points from 1 to n, and join blue and red points having the same numbers. Assume from the contrary that each enumeration does not satisfy the condition of the problem. Consider such enumeration that the total length of the segments is minimal, and let AB and CD be intersecting segments of this numbering. It is easy to check that $AC + AD \leqslant AB + CD$, also as well as $AD + BC \leqslant AB + CD$. Here we come to a contradiction with the minimality of the sum of the segments. $\square$

By $\mathbb{R} = \mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$ denote, respectively, a line, a plane and an 3-dimensional space. A point of $\mathbb{R} = \mathbb{R}^1$ is a real number, a point of $\mathbb{R}^2$ is a pair $\mathbf{x} = (x_1, x_2)$, where $x_1, x_2 \in \mathbb{R}$, and a point of $\mathbb{R}^3$ is a vector $\mathbf{x} = (x_1, x_2, x_3)$, where $x_1, x_2, x_3 \in \mathbb{R}$. An element of $\mathbb{R}^n$ is a point (a vector) $\mathbf{x} = (x_1, \ldots, x_n)$, where $x_i \in \mathbb{R}$, $i = 1,...,n$ (x_i is a coordinate).

A hyperplane h of $\mathbb{R}^d$ is a set of points $\mathbf{x} = (x_1, \ldots, x_d)$ that can be described with a single linear equation of the following form (where at least one of the a_i's is non-zero):

$$\sum_{i=1}^{d} a_i x_i = a_0.$$

Checking understanding. What does mean a hyperplane in $\mathbb{R}^2$ and $\mathbb{R}^3$?

We will say that a hyperplane h *bisects* a finite set A if neither of the two open half-spaces bounded by h contains more than $\lfloor |A|/2 \rfloor$ points of A.

Ham Sandwich Theorem. Every d finite sets $A_1, \ldots, A_d \subset \mathbb{R}^d$ can be simultaneously bisected by a hyperplane.

The ham sandwich theorem takes its name from the case when $d = 3$ and the three objects of any shape are a chunk of ham, a piece of cheese and a chunk of bread — notionally, a sandwich — which can then all be simultaneously bisected with a single cut (i.e., a plane).

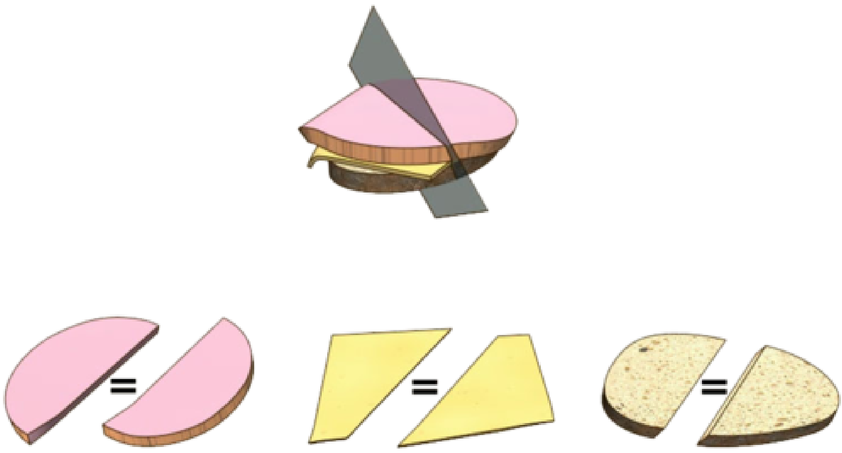


Figure 8: Sandwich cutting

1.16. Consider $3n$ points in general position in the space such that n points are "red", other n points are "blue" and the others are "green". Prove that the points can be portioned into "rainbow" 3-tuples in such a way that no two triangles with vertices on points of triples intersect each other.

Proof. Let A_i, i = 1,2,3, be a set of points of i-th colour. We proceed by induction on n. If $n > 1$ is odd, there is a plane h bisecting each A_i and containing exactly one point of each color. We let the points in h form one 3-tuple and use induction for the subsets in the open half-spaces. For n even, we invoke the ham-sandwich theorem, which provides a plane bisecting all the A_i. Then we should consider each open half-space, and this completes proving of the inductive step. □

1.17.* Two thieves have stolen a precious necklace of nearly immeasurable value, not only because of the precious stones (diamonds, sapphires, rubies, etc.), but also because these are set in pure platinum. The thieves do not know the values of the stones of various kinds, and so they want to divide the stones of each kind evenly. In order to waste as little platinum as possible, they want to achieve this by as few cuts as possible. We assume that the necklace is open (with two ends) and that there are d different kinds of stones, an even number of each kind. How many cuts are necessary to do so?

(a) $(d-1)$ cuts?

(b) d cuts?

Comment. Solve exercises on the Veronese map for better understanding of this problem.

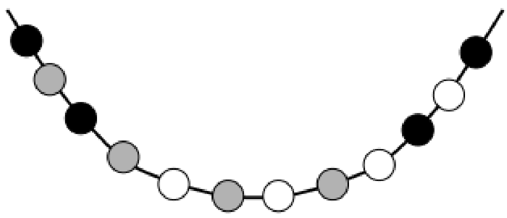


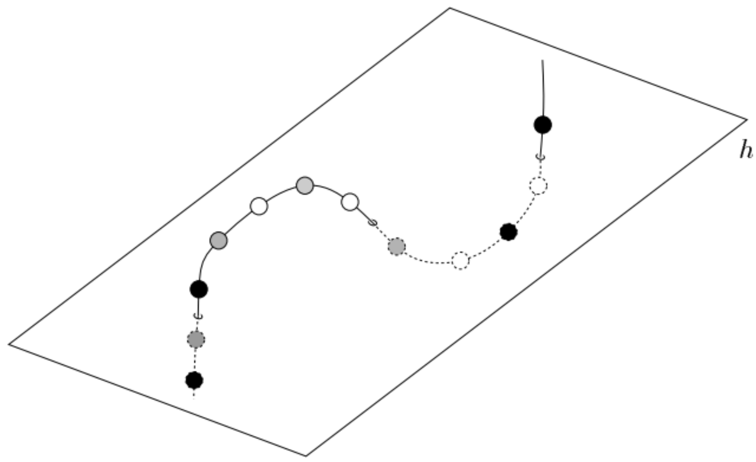
Figure 9: For the necklace here, 3 cuts should suffice

Proof. (a) It is easy to see that at least d cuts may be necessary: place the stones of the first kind first, then the stones of the second kind, and so on.

(b) We place the considered necklace into $\mathbb{R}^d$ along the moment curve. Let $\gamma(t) = (t, t^2, \ldots, t^d)$ be the parametric expression of the moment curve γ. If the necklace has n stones, we define

$$A_i = \{\gamma(k) : \text{ the k-th stone is of the i-th kind, } k = 1, 2, \ldots n\}.$$

Let us also call the points of A_i the stones of the i-th kind. By the ham sandwich theorem, there exists a hyperplane h simultaneously bisecting each A_i. This h cuts the moment curve, and the necklace lying along it, in at most d places. All the sets A_i were assumed to be of even size, so h contains no stones, and these cuts are as required in the necklace problem.



$\square$

2 Geometric Constructions

We will say that a set P of n points and a set $\mathcal{L}$ of n lines in the plane form a *configuration* n_d if, for any line of $\mathcal{L}$, there exist exactly d points of P lying on this line, and, for any point of P, there exist exactly d lines of $\mathcal{L}$ passing through this point. This implies that $I(\mathcal{L},P) = nd$

2.1. Construct an example of configuration 9_3.

Hint: This is a well-known geometric theorem.

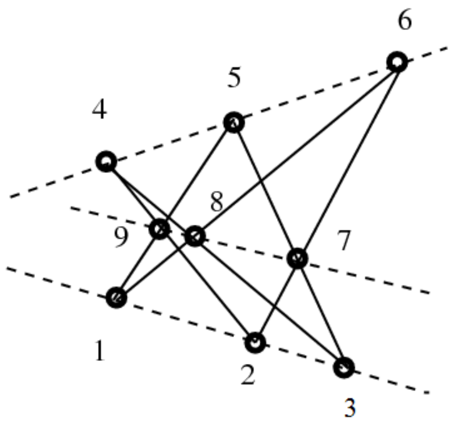
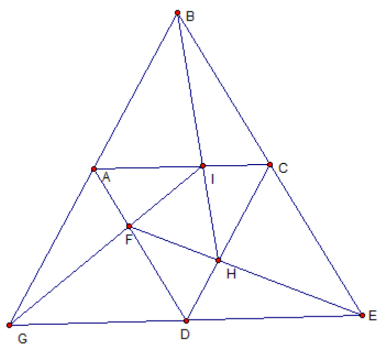


Figure 10: Pappus's 9_3

2.2. Construct an example of configuration 9_3 that is different from the previous one.



2.3. Construct an example of configuration 10_3.

Hint: This is a well-known geometric theorem.

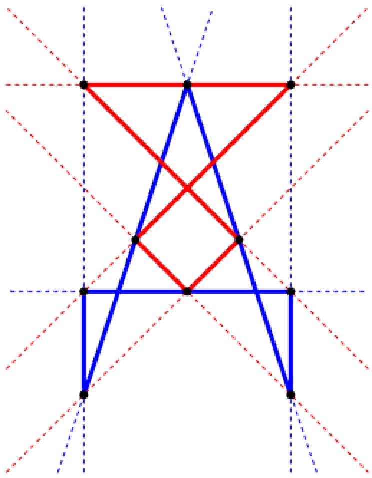
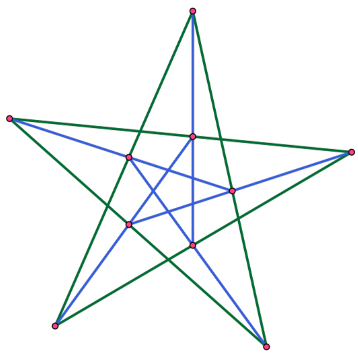


Figure 11: Desargue's 10_3

2.4. Construct an example of configuration 10_3 that is different from the previous one.



2.5.* Prove that there exist a constant factor C such that for any N we can find a set $\mathcal{L}$ of $n, n > N$, lines and a set P of $n, n > N$, points with $I(\mathcal{L}, P) > Cn^{4/3}$.

Hint: Look at the figure 2 on the page 1.

Proof. For simplicity, we suppose that $n = 4k^3$ for a natural k. Let us prove that $I(n, n) \gtrsim n^{4/3}$ for some arrangement (L, P) of points and straight lines to be constructed. For P take the grid $\{0, 1, \dots k\} \times \{0, 1, \dots 4k^2 - 1\}$. For L take all lines $y = ax + b$, with $(a, b) \in \{0, 1, \dots 2k - 1\} \times \{0, 1, \dots 2k^2 - 1\}$. Then for $x \in [0; k)$ one has $ax + b < ak + b < 2k^2 + 2k^2 = 4k^2$, so for each $i = 0; \dots k_i - 1$ each line contains a point of P with $x = i \in \{0, 1, \dots k\}$. Thus $I(L, P) \approx k^4 \approx n^{4/3}$ $\square$

3 Algebraic Geometry

Let f and g be two functions defined on some subset of the real numbers. One writes $g(x) = O(f(x))$ if and only if there exists a positive constant $C > 0$ such that $|g(x)| \leqslant Cf(x)$ for all x.

Given a polynomial $f(x, y) = \sum_{i,j \leqslant 0} a_{i,j} x^i y^j$. The *zero* set Z_f of $f(x, y)$ is the subset of points (x, y) in $\mathbb{R}^2$ on which $f(x, y) = 0$. The polynomial $f(x,y)$ is of degree p ($\deg f = p$) if $p = \max\{i + j \mid a_{i,j} \neq 0\}$.

A polynomial $f(x,y)$ is called a *monomial* if $a_{i,j} = 1$ for exactly one pair (i,j) and $a_{i,j} = 0$ otherwise (roughly speaking, a polynomial which has only one term).

3.1. Consider $f(x, y)$ of degree d and an arbitrary line l. Prove that either $l \in Z_f$, or $|l \cap Z_f| \leqslant D$.

Proof. Writing l in parametric form $\{(u_1 t + v_1,\, u_2 t + v_2) \mid t \in \mathbb{R}\}$, we get that the points of $l \cap Z(f)$ are roots of the univariate polynomial $g(t) := f(u_1 t + v_1, u_2 t + v_2)$, which is of degree at most D. Thus, either g is identically 0, or it has at most D roots. □

3.2. Consider a polynomial $f(x, y)$ of degree d. Prove that the number of distinct lines contained in the set Z_f does not exceed d.

Proof. We need to know that a nonzero bivariate polynomial (i.e., with at least one nonzero coefficient) does not vanish on all of $\mathbb{R}^2$. Now we fix a point $P \in \mathbb{R}^2$ not belonging to $Z(f)$. Let us suppose that $Z(f)$ contains lines $l_1, \dots l_k$. We choose another line l passing through P that is not parallel to any l_i and not passing through any of the intersections $l_i \cap l_j$. (Such an l exists since only finitely many directions need to be avoided.) Then l is not contained in $Z(f)$ and it has k intersections with $\cup_{i=1}^{k} l_i$. A result of the previous problem yields $k \leqslant D$. □

3.3. Show that the number of bivariate monomials of degree at most d equals $\binom{d+2}{2}$.

Hint. In other words, we must find the number of pairs (i,j) of nonnegative integers with $i + j \leqslant D$.

Let P be a set of n points in the plane, and let r be a parameter, $1 < r \leqslant n$. We say that $f(x, y)$ is an *r-partitioning polynomial* for P if no connected component of $\mathbb{R}^2 \setminus Z_f$ contains more than n/r points of A.

In the sequel, we will sometimes call the connected components of $\mathbb{R}^2\setminus Z(f)$ cells. Let us also stress that the cells are open sets. The points lying on $Z(f)$ do not belong to any cell.

3.4. For any $r > 1$, every finite point set admits an r-partitioning polynomial of degree at most r.

Hint. Construct r parallel lines dividing the given set into equal parts.

Definition. Given an arbitrary integer d, and let $D = \binom{d+2}{2} - 1$. A map $\varphi : \mathbb{R}^2 \to \mathbb{R}^D$ is said to be the *Veronese map of degree d* if this map is given by the following formula:

$$\varphi(x,y) := (x^i y^j)_{(i,j)|1\leqslant i+j\leqslant d} \in \mathbb{R}^D.$$

(We think of the coordinates in $\mathbb{R}^l$ as indexed by pairs (i,j) with $1 \leqslant i+j \leqslant D$.)

Note that we can apply the Veronese map of degree 2 to prove that any 5 points in the plane such that no 4 points are collinear uniquely determine a conic passing through the given points (A *c*onic is a zero set of a bivariate polynomial of degree 2). In order to check it, we should consider an appropriate hyperplane containing the images of the given points under the Veronese map (We have D=5 in the considered case, and one can show that there exists a unique hyperplane on which all five images of the points lie). It is easy to check that the conic in the plane which corresponds the hyperplane is required.

3.5. Given finite sets $A_1, \ldots, A_k$. Let k be an integer such that $\binom{k+2}{2} - 1 \geqslant l$. Prove that there exists a polynomial of degree at most k that is an 2-partitioning polynomial for every one of these sets.

Proof. Problem 3.3 yields that $\binom{D+2}{2}$ is the number of monomials in a bivariate polynomial of degree at most D. We set $l := \binom{D+2}{2} - 1$, and we let $\phi : \mathbb{R}^2 \to \mathbb{R}^l$ denote the *Veronese map*, given by

$$\phi(x,y) := (x^i y^j)_{(i,j)|1\leqslant i+j\leqslant D} \in \mathbb{R}^l.$$

(We think of the coordinates in $\mathbb{R}^l$ as indexed by pairs (i,j) with $1 \leqslant i+j \leqslant D$.)

We set $A_i' := \phi(A_i)$, $i = 1,2,\ldots,k$, and by ham sandwich theorem, we let h be a hyperplane simultaneously bisecting $A_1',\ldots,A_k'$. Then h has an equation of the form $a_{00} + \sum\limits_{i,j} a_{i\,j} z + i\,j = 0$, where $(z_{i\,j})_{(i,j)|1\leqslant i+j\leqslant d}$ are the coordinates in $\mathbb{R}^k$. It is easy to check that $f(x,y) := \sum_{i,j} a_{i,j} x^i y^j$ is the desired polynomial (where here the sum includes a_{00} too). $\square$

3.6. For every $r > 1$, every finite point set $P \subset \mathbb{R}^2$ admits an r-partitioning polynomial f of degree at most $O(\sqrt{r})$.

Proof. We inductively construct collections $\mathcal{P}_0, \mathcal{P}_1, \ldots$, each consisting of disjoint subsets of P, such that $|\mathcal{P}_j| \leqslant 2^j$ for each j. We start with $\mathcal{P}_0 := \{P\}$. Having constructed $\mathcal{P}_j$, we use the previous problem to construct a polynomial f_j, of degree $deg(f_j) \leqslant \sqrt{2 \cdot 2^j}$, that bisects each of the sets of $\mathcal{P}_j$. Then for every subset $Q \in \mathcal{P}_j$, we let Q^+ consist of the points of Q at which $f_j > 0$, and let Q^- consist of the points of Q with $f_j < 0$, and we put $\mathcal{P}_{j+1} := \cup_{Q \in \mathcal{P}_j} \{Q^+, Q^-\}$.

Each of the sets in $\mathcal{P}_j$ has a size at most $|P|/2^j$. We let $t = \lceil \log 2r \rceil$; then each of the sets in $\mathcal{P}_t$ has a size at most $|P|/r$. We set $f := f_1 f_2 \ldots f_t$. By the construction, no component of $\mathbb{R}^2 \setminus Z(f)$ can contain points of two different sets in $\mathcal{P}_t$, because any arc connecting a point in one subset to a point in another subset must contain a point at which one of the polynomials f_j vanishes, so the arc must cross $Z(f)$. Thus, f is an r-partitioning polynomial for P. It remains to bound the degree:

$$deg(f) = deg(f_1) + deg(f_2) + \cdots + deg(f_t) \leqslant \sqrt{2} \sum_{j=1}^{t} 2^{j/2} \leqslant \frac{2}{\sqrt{2} - 1} 2^{t/2} \leqslant c\sqrt{r},$$

where $c = 2\sqrt{2}/(\sqrt{2} - 1) < 7$. □

4 The First Proof of the Szemeredi-Trotter Theorem

4.1. Prove that $I(m,n) = I(n,m)$.

Proof. Applying the polar transformation, lines are replaced by points and points are replaced by lines. Notice that a 'line-point' incidence enters to a 'point-line' incidence (as well as a line which has no incidence with a point goes to the point not incidence to the corresponding line). Thus, if we have a set of points and lines having $I(n,m)$ incidences, then we can transformate it to the set of points and lines, which has the same number of incidences. This yields $I(m,n) \geqslant I(n,m)$. Similarly, we can get the converse inequality. $\square$

If we apply a bound in the problem 1.8, then we will not get a proof of the Szemeredi-Trotter theorem immediately.

In what follows, we assume $|L| = |P| = n$. First of all, we construct an r-partitioning polynomial $f(x,y)$ of minimal degree for the given set P.

By $L_0 \subset L$ denote a subset of lines such that $l \in L$ and $l \subset Z_f$, by $P_0 \subset P$ denote a set of points $p \in P \cap Z_f$. Suppose Z_f divides the plane into s parts. By P_i denote points of P lying in the i-th part of the plane, and by L_i denote lines of $\mathcal{L}$ passing through the i-th part of the plane.

4.2. Prove that there exist constants C_1, C_2, C_3 such that

a) $I(L_0, P_0) \leqslant C_1 n\sqrt{r}$;

b) $I(L \setminus L_0, P_0) \leqslant C_2 n\sqrt{r}$;

c) $\sum\limits_{i=1}^{s} I(L_i, P_i) \leqslant C_3(n\sqrt{r} + n^2/r)$.

Proof. a) Since the number of distinct lines contained in Z_f is at most $\deg f \leqslant C_1\sqrt{r}$, where C_1 is a certain constant (see problems **3.2** and **3.6**), and since the number of points in P_0 does not exceed n, then by trivial bound, we obtain $I(L_0, P_0) \leqslant C_1 n\sqrt{r}$.
b) Since each line of $L \setminus L_0$ intersects Z_f in at most $\deg f$ points, and since $\deg f \leqslant C_2\sqrt{r}$, then $I(L \setminus L_0, P_0) \leqslant C_2 n\sqrt{r}$.

c) By the Problem **1.8 (a)** we get

$$\sum_{i=1}^{s} I(L_i, P_i) \leqslant \sum_{i=1}^{s} \left(|L_i| + |P_i|^2\right)$$

Let $D = deg(f) = O(\sqrt{r})$. We have

$$\sum_{i=1}^{s} |L_i| = O((D+1)n) = O(\sqrt{r}n),$$

since no line intersects more than $D+1$ of the sets P_i. Finally,

$$\sum_{i=1}^{s} |P_i|^2 \leqslant (\max_i |P_i|) \cdot \sum_{i=1}^{s} |P_i| = O(n^2/r).$$

$\square$

4.3. Having chosen a certain r, prove the Szemeredi-Trotter theorem.

Proof. Setting $r = n^{2/3}$, we obtain the proof of the Szemeredi-Trotter theorem for the case $n = m$. $\square$

4.4. Prove the the Szemeredi-Trotter theorem in the general case:
The Szemeredi-Trotter theorem. $I(n,m) = O((nm)^{2/3} + n + m)$.

Proof. We generalize the proof for an arbitrary m as follows. We may assume, without loss of generality, that $m \leqslant n$; the complementary case is handled by interchanging the roles of P and L, via a standard planar duality. We may also assume that $\sqrt{n} \leqslant m$, since otherwise, the theorem follows from Problem **1.8**. Then we set $r := \frac{m^{4/3}}{n^{2/3}}$. Noting that $1 \leqslant r \leqslant m$ for the assumed range of m, we then proceed as in the case $m = n$ above. We get $D = deg(f) = O(m^{2/3}n^{1/3})$, and we check that all the partial bounds in the proof are at most $O(m^{2/3}n^{2/3})$. $\square$

5 Applications of the Szemeredi-Trotter Theorem

5.1. (a) Given a finite set P, $|P| = m$, in the plane. Prove that the number of lines passing through k distinct points of P is at most $C(m^2/k^3 + m/k)$, where C is some constant, which does not depend on m, k.

(b) Prove that such lines define at most $D(m^2/k^2 + m)$ incidences with P, where D is some constant, which does not depend on m, k.

5.2. Let P be a set of points in the plane and let L be the set of lines containing at least 2 points in P. Then there exist such $c_1, c_2 > 0$ that one of these two cases must hold:

1. There exists a line in L that contains at least $c_1|P|$ points.
2. $|L| > c_2|P|^2$.

5.3. Let $A \subset \mathbb{R}$ be a finite set. Then

$$\max\{|A+A|; |A \cdot A|\} \geqslant |A|^{5/4}.$$

Here

$$A+A = \{a_1 + a_2 : a_1 \in A, a_2 \in A\}, A \cdot A = \{a_1 \cdot a_2 : a_1 \in A, a_2 \in A\}.$$

Solutions. See [2], pp.15–17, [7].

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