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Universal cycles for permutations

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Abstract

A universal cycle for permutations is a word of length $n!$ such that each of the $n!$ possible relative orders of n distinct integers occurs as a cyclic interval of the word. We show how to construct such a universal cycle in which only $n + 1$ distinct integers are used. This is best possible and proves a conjecture of Chung, Diaconis and Graham.

Keywords: Universal cycles; Combinatorial generation; Permutations

1. Introduction

A *de Bruijn cycle* of order n is a word in $\{0, 1\}^{2^n}$ in which each n-tuple in $\{0, 1\}^n$ appears exactly once as a cyclic interval (see [2]). The idea of a universal cycle generalizes the notion of a de Bruijn cycle.

Suppose that $\mathcal{F}$ is a family of combinatorial objects with $|\mathcal{F}| = N$, each of which is represented (not necessarily in a unique way) by an n-tuple over some alphabet A. A *universal cycle* (or *ucycle*) for $\mathcal{F}$ is a word $u_1u_2 \dots u_N$ with each $F \in \mathcal{F}$ represented by exactly one $u_{i+1}u_{i+2} \dots u_{i+n}$ where, here and throughout, index addition is interpreted modulo N. With this terminology a de Bruijn cycle is a ucycle for words of length n over $\{0, 1\}$ with a word represented by itself. The definition of ucycle was introduced by Chung, Diaconis and Graham in [1]. Their paper, and the references therein, forms an good overview of the topic of universal cycles. The cases considered by them include $\mathcal{F}$ being the set of permutations of an n-set, r-subsets of an n-set, and partitions of an n-set.

In this paper we will be concerned with ucycles for permutations: our family $\mathcal{F}$ will be S_n, which we will regard as the set of all n-tuples of distinct elements of $[n] = \{1, 2, \dots n\}$. It is not immediately obvious how we should represent permutations with words. The most natural thing to do would be to take $A = [n]$ and represent a permutation by itself, but it is easily verified that (except when $n \leq 2$) it is not possible to have a ucycle in this case. Indeed, if every cyclic interval of a word is to represent a permutation then our word must repeat with period n, and so only n distinct permutations can be represented. Another possibility which we mention in passing would be to represent the permutation $a_1a_2 \dots a_n$ by $a_1a_2 \dots a_{n-1}$. It is clear that the permutation is determined by this. It was shown by Jackson [3] (using similar techniques to those used for de Bruijn cycles) that these ucycles exist for all n. Recently an efficient algorithm for constructing such ucycles was given by Williams [4]. He introduced the term *shorthand universal cycles for permutations* to describe them. Alternatively, Chung, Diaconis and Graham in [1] consider ucycles

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for permutations using a larger alphabet where each permutation is represented by any n-tuple in which the elements have the same relative order. Our aim is to prove their conjecture that such ucycles always exist when the alphabet is of size $n+1$, the smallest possible. In contrast to the situation with shorthand universal cycles, the techniques used for de Bruijn cycles do not seen to help with this so a different approach is needed.

To describe the problem more formally we need the notion of order-isomorphism. If $a = a_1 a_2 \dots a_n$ and $b = b_1 b_2 \dots b_n$ are n-tuples of distinct integers, we say that a and b are *order-isomorphic* if

$$a_i < a_j \Leftrightarrow b_i < b_j$$

for all $1 \le i, j \le n$. Note that no two distinct permutations in S_n are order-isomorphic, and that any n-tuple of distinct integers is order-isomorphic to exactly one permutation in S_n. Hence, the set of n-tuples of distinct integers is partitioned into $n!$ order-isomorphism classes which correspond to the elements of S_n.

We say that a word $u_1 u_2 \dots u_{n!}$ over an alphabet $A \subset \mathbb{Z}$ is a ucycle for S_n if there is exactly one $u_{i+1} u_{i+2} \dots u_{i+n}$ order-isomorphic to each permutation in S_n. For example 012032 is a ucycle for S_3. Let $M(n)$ be the smallest integer m for which there is a ucycle for S_n with $|A| = m$. Note that if $|A| = n$ then each permutation is represented by itself and so, as we noted earlier, no ucycle is possible (unless $n \le 2$). We deduce that $M(n) \ge n+1$ for all $n \ge 3$. Chung, Diaconis and Graham in [1] give the upper bound $M(n) \le 6n$ and conjecture that $M(n) = n+1$ for all $n \ge 3$. Our main result is that this conjecture is true.

Theorem 1. *For all $n \ge 3$ there exists a word of length $n!$ over the alphabet $\{0, 1, 2, \dots, n\}$ such that each element of S_n is order-isomorphic to exactly one of the $n!$ cyclic intervals of length n.*

We prove this by constructing such a word inductively. The details of our construction are in the next section. Having shown that such a word exists, it is natural to ask how many there are. In the final section we give some bounds on this.

Our construction works for $n \ge 5$. For smaller values of n it is a relatively simple matter to find such words by hand. For completeness examples are 012032 for $n = 3$, and 012301423042103421302143 for $n = 4$.

2. A construction of a universal cycle

We will show how to construct a word of length $n!$ over the alphabet $\{0, 1, 2, \dots, n\}$ such that for each $a \in S_n$ there is a cyclic interval which is order-isomorphic to a.

Before describing the construction we make a few preliminary definitions.

As is standard for universal cycle problems we let $G_n = (V, E)$, the *transition graph*, be the directed graph with

$$V = \{(a_1 a_2 \dots a_n) : a_i \in \{0, 1, 2, \dots, n\}, \text{ and } a_i \ne a_j \text{ for all } i \ne j\}$$
$$E = \{(a_1 a_2 \dots a_n)(b_1 b_2 \dots b_n) : a_{i+1} = b_i \text{ for all } 1 \le i \le n-1\}.$$

Notice that every vertex of G_n has out-degree and in-degree both equal to 2.

The vertices on a directed cycle in G_n plainly correspond to the n-tuples which occur as cyclic intervals of some word. Our task, therefore, is to find a directed cycle in G_n of length $n!$ such that for each $a \in S_n$ there is some vertex of our cycle which is order-isomorphic to a. This is in contrast to many universal cycle problems where we seek a Hamilton cycle in the transition graph.

We define the map on the integers:

$$s_x(i) = \begin{cases} i & \text{if } i < x \\ i+1 & \text{if } i \ge x. \end{cases}$$

We also, with a slight abuse of notation, write s_x for the map constructed by applying this map coordinatewise to an n-tuple. That is,

$$s_x(a_1 a_2 \dots a_n) = s_x(a_1) s_x(a_2) \dots s_x(a_n).$$

The point of this definition is that if $a = a_1 a_2 \dots a_n \in S_n$ is a permutation of $[n]$ and $x \in [n+1]$ then $s_x(a)$ is the unique n-tuple of elements of $[n+1] \setminus \{x\}$ which is order-isomorphic to a. Note that, as will become clear, this is the definition we need even though our final construction will produce a ucycle for permutations of $[n]$ using alphabet $\{0, 1, 2, \dots, n\}$.

We also define a map r on n-tuples which permutes the elements of the n-tuple cyclically. That is,

$$r(a_1 a_2 \dots a_n) = a_2 a_3 \dots a_{n-1} a_n a_1.$$

Note that $(a, r(a))$ is an edge of G_n and that $r^n(a) = a$.

As indicated above, we prove Theorem 1 by constructing a cycle of length $n!$ in G_n such that for each $a \in S_n$ the cycle contains a vertex which is order-isomorphic to a. Our approach is to find a collection of short cycles in G_n which between them contain one vertex from each order-isomorphism class and to join them up. The joining up of the short cycles requires a slightly involved induction step which is where the main work lies.

Proof of Theorem 1. *Step* 1: Finding short cycles in G_n.

The first step is to find a collection of short cycles (each of length n) in G_n which between them contain exactly one element from each order-isomorphism class of S_n. These cycles will use only n elements from the alphabet and we will think of each cycle as being "labelled" with the remaining unused element. Suppose that for each $a = a_1 a_2 \dots a_{n-1} \in S_{n-1}$ we choose a label $l(a)$ from $[n]$. Let $0a$ be the n-tuple $0a_1 a_2 \dots a_{n-1}$. We have the following cycle in G_n:

$$s_{l(a)}(0a), r(s_{l(a)}(0a)), r^2(s_{l(a)}(0a)), \dots, r^{n-1}(s_{l(a)}(0a)).$$

We denote this cycle by $\mathcal{C}(a, l(a))$. As an example, $\mathcal{C}(42135, 2)$ is the following cycle in G_6:

$$053146 \to 531460 \to 314605 \to 146053 \to 460531 \to 605314 \to 053416,$$

where arrows denote directed edges of G_6.

Note that for any choice of labels (that is any map l) the cycles $\mathcal{C}(a, l(a))$ and $\mathcal{C}(b, l(b))$ are disjoint when $a, b \in S_n$ are distinct. Consequently, whatever the choice of labels, the collection of cycles

$$\bigcup_{a \in S_{n-1}} \mathcal{C}(a, l(a))$$

is a disjoint union. It is easy to see that the vertices on these cycles contain between them exactly one n-tuple order-isomorphic to each permutation in S_n.

We must now show how, given a suitable labelling, we can join up these short cycles.

Step 2: Joining two of these cycles

Suppose that $\mathcal{C}_1 = \mathcal{C}(a, x)$ and $\mathcal{C}_2 = \mathcal{C}(b, y)$ are two of the cycles in G_n described above. What conditions on a, b and their labels x, y will allow us to join these cycles?

We may assume that $x \le y$. Suppose further that $1 \le x \le y - 2 \le n - 1$, and that a and b satisfy the following:

$$b_i = \begin{cases} a_i & \text{if } 1 \le a_i \le x - 1 \\ a_i + 1 & \text{if } x \le a_i \le y - 2 \\ x & \text{if } a_i = y - 1 \\ a_i & \text{if } y \le a_i \le n - 1. \end{cases}$$

If this happens we will say that the pair of cycles $\mathcal{C}(a, x), \mathcal{C}(b, y)$ are *linkable*.

In this case $s_x(0a)$ and $s_y(0b)$ agree at all but one position; they differ only at the t for which $a_t = y - 1$ and $b_t = x$. It follows that there is a directed edge in G_n from

$$r^t(s_x(0a)) = s_x(a_t \dots a_{n-1} 0 a_1 \dots a_{t-1})$$

to

$$r^{t+1}(s_y(0b)) = s_y(b_{t+1} \dots b_{n-1} 0 b_1 \dots b_t).$$

Similarly, there is a directed edge in G_n from

$$r^t(s_y(0b)) = s_x(b_t \dots b_{n-1} 0 b_1 \dots b_{t-1})$$

to

$$r^{t+1}(s_x(0a)) = s_y(a_{t+1} \dots a_{n-1} 0 a_1 \dots a_t).$$

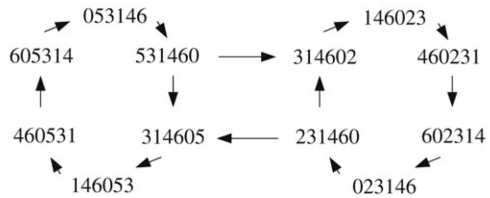

Fig. 1. The cycles $\mathcal{C}(42135, 2)$ and $\mathcal{C}(23145, 5)$ are linkable.

If we add these edges to $\mathcal{C}_1 \cup \mathcal{C}_2$ and remove the edges

$$r^t(s_x(0a))r^{t+1}(s_x(0a))$$

and

$$r^t(s_y(0b))r^{t+1}(s_y(0b)),$$

then we produce a single cycle of length $2n$ whose vertices are precisely the vertices in $\mathcal{C}_1 \cup \mathcal{C}_2$.

We remark that if $x = y - 1$ then the other conditions imply that $a = b$ and so although we can perform a similar linking operation it is not useful. If $x = y$ then b is not well defined.

As an example of the linking operation consider the linkable pair of 6-cycles $\mathcal{C}(42135, 2)$ and $\mathcal{C}(23145, 5)$ in G_6. If we add the edges $531460 \to 314602$ and $231460 \to 314605$, and remove the edges $531460 \to 314605$ and $231460 \to 314602$ then a single cycle of length 12 in G_6 is produced. These cycles and the linking operation are shown in Fig. 1.

Step 3: Joining all of these cycles

We now show that this linking operation can be used repeatedly to join a collection of disjoint short cycles, one for each $a \in S_n$, together.

Let $H_n = (V, E)$ be the (undirected) graph with,

$$V = \{(a, x) : a \in S_{n-1}, x \in [n]\}$$
$$E = \{(a, x)(b, y) : \mathcal{C}(a, x), \mathcal{C}(b, y) \text{ are linkable}\}.$$

If we can find a subtree T_n of H_n of order $(n-1)!$ which contains exactly one vertex (a, x) for each $a \in S_{n-1}$ then we will be able to construct the required cycle. Take any vertex $(a, l(a))$ of T_n and consider the cycle $\mathcal{C}(a, l(a))$ associated with it. Consider also the cycles associated with all the neighbours in T_n of $(a, l(a))$. The linking operation described above can be used to join the cycles associated with these neighbours to $\mathcal{C}(a, l(a))$. This is because the definition of adjacency in H_n guarantees that we can join each of these cycles individually. Also, the fact that every vertex in G_n has out-degree 2 means that the joining happens at different places along the cycle. That is if $(b, l(b))$ and $(c, l(c))$ are distinct neighbours of $(a, l(a))$ then the edge of $\mathcal{C}(a, l(a))$ which must be deleted to join $\mathcal{C}(b, l(b))$ to it is not the same as the one which must be deleted to join $\mathcal{C}(c, l(c))$ to it. We conclude that we can join all of the relevant cycles to the cycle associated with $(a, l(a))$. The connectivity of T_n now implies that we can join all of the cycles associated with vertices of T_n into one cycle. This is plainly a cycle with the required properties.

The next step is to find such a subtree in H_n.

Step 4: Constructing a Suitable Tree

We will prove, by induction on n, the stronger statement that for all $n \geq 5$, there is a subtree T_n of H_n of order $(n-1)!$ which satisfies:

1. for all $a \in S_{n-1}$ there exists a unique $x \in [n]$ such that $(a, x) \in V(T_n)$,
2. $(12\ldots(n-1), 1) \in V(T_n)$,
3. $(23\ldots(k-1)1(k)(k+1)\ldots(n-1), k) \in V(T_n)$ for all $3 \leq k \leq n$,
4. $(32145\ldots(n-1), 2) \in V(T_n)$,
5. $(243156\ldots(n-1), 3) \in V(T_n)$,
6. $v(31245\ldots(n-1))$ is a leaf in T_n,
7. $v(24135\ldots(n-1))$ is a leaf in T_n.

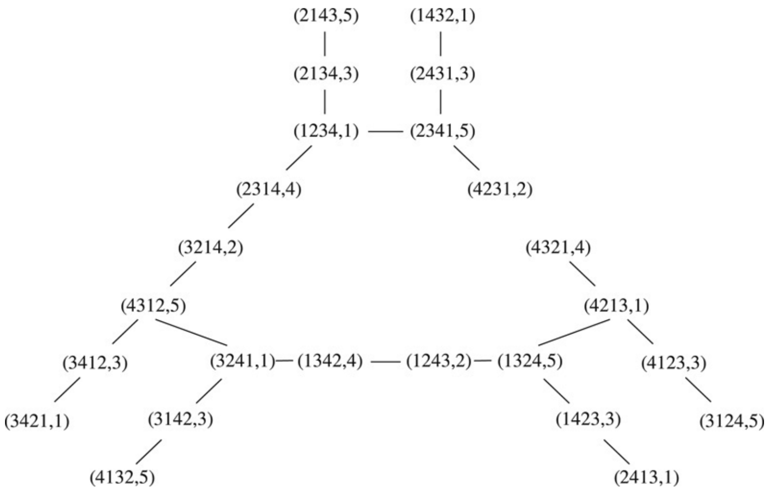

Fig. 2. A suitable choice for T_5.

Where, for a tree satisfying property 1, we denote the unique vertex in $V(T_n)$ of the form (a, x) by $v(a)$.

For $n = 5$ a suitable tree can be found. One such is given in Fig. 2.

Suppose that $n \geq 5$ and that we have a subtree T_n of the graph H_n which satisfies the above conditions. We will use this to build a suitable subtree of H_{n+1}.

A key observation for our construction is that the map from $V(H_n)$ to $V(H_{n+1})$ obtained by replacing each vertex $(a_1a_2 \ldots a_{n-1}, x) \in V(H_n)$ by $(1(a_1+1)(a_2+1)\ldots(a_{n-1}+1), x+1) \in V(H_{n+1})$ preserves adjacency. It follows that subgraphs of H_n are mapped into isomorphic copies in H_{n+1} by this map. Further, applying a fixed permutation to the coordinates of the n-tuple associated with each vertex of H_{n+1} gives an automorphism of H_{n+1} and so subgraphs of H_{n+1} are mapped into isomorphic copies.

We take n copies of T_n. These copies will be modified to form the building blocks for our subtree of H_{n+1} as follows.

In the first copy we replace each vertex $(a_1a_2 \ldots a_{n-1}, x)$ by

$$(1(a_3+1)(a_1+1)(a_4+1)(a_2+1)(a_5+1)(a_6+1)\ldots(a_{n-1}+1), x+1).$$

By the observation above this gives an isomorphic copy of T_n in H_{n+1}. We denote this tree by $T_{n+1}^{(0)}$.

In the next copy we replace each vertex $(a_1a_2 \ldots a_{n-1}, x)$ by

$$((a_3+1)1(a_2+1)(a_1+1)(a_4+1)(a_5+1)\ldots(a_{n-1}+1), x+1).$$

We denote this tree by $T_{n+1}^{(1)}$.

For all $2 \leq k \leq n-1$, we take a new copy of T_n and modify it as follows. We replace each vertex $(a_1, a_2 \ldots a_{n-1}, x)$ by

$$((a_k+1)(a_1+1)(a_2+1)\ldots(a_{k-1}+1)1(a_{k+1}+1)\ldots(a_{n-1}+1), x+1).$$

We denote these trees by $T_{n+1}^{(2)}, T_{n+1}^{(3)}, \ldots, T_{n+1}^{(n-1)}$.

As we mentioned this results in n subtrees of H_{n+1}. They are clearly disjoint because the position in which 1 appears in the first coordinate of each vertex is distinct for distinct trees. Let F_{n+1} be the n component subforest of H_{n+1} formed by taking the union of the trees $T_{n+1}^{(k)}$ for $0 \leq k \leq n-1$.

It is also easy to see that for every $a \in S_n$ there is a vertex in F_{n+1} of the form (a, x) for some $x \in [n+1]$. It remains to show that the n components can be joined up to form a single tree with the required properties.

Notice that it is a consequence of the construction of the $T_{n+1}^{(k)}$ that if $v(l_1l_2 \ldots l_{n-1})$ is a leaf in T_n then the following vertices are all leaves in F_{n+1}:

- $v(1(l_3+1)(l_1+1)(l_4+1)(l_2+1)(l_5+1)(l_6+1)\ldots(l_{n-1}+1))$
- $v((l_3+1)1(l_2+1)(l_1+1)(l_4+1)(l_5+1)\ldots(l_{n-1}+1))$
- $v((l_k+1)(l_1+1)(l_2+1)\ldots(l_{k-1}+1)1(l_{k+1}+1)\ldots(l_{n-1}+1))$ for $2 \leq k \leq n-1$.

We claim that $v(12\dots n)$ is a leaf in $T_{n+1}^{(0)}$, and hence a leaf in F_{n+1}. This follows from the fact that $v(24135\dots(n-1))$ is a leaf in T_n and the remark above. We delete this vertex from F_{n+1} to form a new forest.

The construction of the $T_{n+1}^{(k-1)}$ and the fact that $(23\dots(k-1)1(k)(k+1)\dots(n-1),k)\in V(T_n)$ for all $3\le k\le n$ means that $(23\dots k1(k+1)(k+2)\dots n, k+1)\in V(F_{n+1})$ for all $3\le k\le n$. Further, the construction of the $T_{n+1}^{(1)}$ and the fact that $(32145\dots(n-1),2)\in V(T_n)$ means that $(2134\dots n,3)\in V(F_{n+1})$.

We add to F_{n+1} a new vertex $(123\dots n,1)$ (this replaces the deleted vertex from $T_{n+1}^{(0)}$) and edges from this new vertex to $(23\dots k1(k+1)(k+2)\dots n,k+1)$ for all $2\le k\le n$. The previous observation shows that all of these vertices are in F_{n+1}, and it is easy to check, using the definition of linkability, that the added edges are in H_{n+1}. This new forest has only two components.

Similarly, the inductive hypothesis that $v(31245\dots(n-1))$ is a leaf in T_n gives that $v(342156\dots n)$ is a leaf in F_{n+1} (using the case $k=3$ of the observation on leaves). We delete this leaf from the forest, replace it with a new vertex $(342156\dots n,1)$, and add edges from this new vertex to $(341256\dots n,3)$ and $(143256\dots n,4)$. The construction of $T_{n+1}^{(2)}$ and the fact that $(32145\dots(n-1),2)\in V(T_n)$ ensures that the first of these vertices is in F_{n+1}. The construction of $T_{n+1}^{(0)}$ and the fact that $(243156\dots(n-1),3)\in V(T_n)$ ensures that the second of these vertices is in our modified $T_{n+1}^{(0)}$.

These modifications to F_{n+1} produce a forest of one component—that is a tree. Denote this tree by T_{n+1}. We will be done if we can show that T_{n+1} satisfies the properties demanded.

Plainly, T_{n+1} contains exactly one vertex of the form (a,x) for each $a\in S_n$. By construction $(12\dots n,1)$, and $(23\dots t1(t+1)(t+2)\dots n,t+1)$ are vertices of T_{n+1} for all $2\le t\le n$. Hence the first three properties are satisfied.

The construction of $T_{n+1}^{(2)}$ and the fact that $(123\dots(n-1),1)\in V(T_n)$ ensures that $(32145\dots n,2)$ is a vertex of T_{n+1}. The construction of $T_{n+1}^{(3)}$ and the fact that $(32145\dots(n-1),2)\in V(T_n)$ ensures that $(243156\dots n,3)$ is a vertex of T_{n+1}. Hence properties 4 and 5 are satisfied.

Finally, the construction of $T_{n+1}^{(1)}$ and the fact that $v(31245\dots(n-1))$ is a leaf in T_n ensures that $v(31245\dots n)$ is a leaf in F_{n+1}. The modifications which F_{n+1} undergoes do not change this and so it is a leaf in T_{n+1}. The construction of $T_{n+1}^{(2)}$ and the fact that $v(31245\dots(n-1))$ is a leaf in T_n ensures that $v(241356\dots n)$ is a leaf in F_{n+1}. Again, the modifications to F_{n+1} do not change this and so this vertex is still a leaf in T_{n+1}. Hence properties 6 and 7 are satisfied.

We conclude that the tree T_{n+1} has the required properties. This completes the construction. □

3. Bounds on the number of universal cycles

Having constructed a ucycle for S_n over the alphabet $\{0,1,\dots,n\}$ it is natural to ask how many such ucycles exist. We will regard words which differ only by a cyclic permutation as the same so we normalize our universal cycles by insisting that the first n entries give a word order-isomorphic to $12\dots n$. We denote by $U(n)$ the number of words of length $n!$ over the alphabet $\{0,1,2,\dots,n\}$ which contain exactly one cyclic interval order-isomorphic to each permutation in S_n and for which the first n entries form a word which is order-isomorphic to $12\dots n$. There is a natural upper bound which is essentially exponential in $n!$ based on the fact that if we are writing down the word one letter at a time we have 2 choices for each letter. We can also show that there is enough choice in the construction of the previous section to prove a lower bound which is exponential in $(n-1)!$. It is slightly surprising that our construction gives a lower bound which is this large. However, the upper and lower bounds are still far apart and we have no idea where the true answer lies.

Theorem 2.

$$420^{\frac{(n-1)!}{24}}\le U(n)\le(n+1)2^{n!-n}.$$

Proof. Suppose we write down our universal cycle one letter at a time. We must start by writing down a word of length n which is order-isomorphic to $12\dots n$; there are $n+1$ ways of doing this. For each of the next $n!-n$ entries we must not choose any of the previous $(n-1)$ entries (all of which are distinct) and so we have 2 choices for each entry. This gives the required upper bound.

Now for the lower bound. We will give a lower bound on the number of subtrees of H_n which satisfies the conditions of step 4 of the proof of Theorem 1. It can be checked that if a universal cycle comes from a subtree of H_n in the way described then the tree is determined by the universal cycle. It follows that the number of such trees is a lower bound for $U(n)$.

Notice that if we have t_n such subtrees of H_n then we have at least t_n^n such subtrees of H_{n+1}. This is because in our construction we took n copies of T_n to build T_{n+1} from and each different set of choices yields a different tree. We conclude that the number of subtrees of H_n satisfying the conditions is at least

$$t_5^{5\times 6\times\cdots\times n-1} = t_5^{\frac{(n-1)!}{24}}.$$

Finally, we bound t_5. We modify the given T_5 be adding edges from (1432, 1) to (2143, 5), from (3421, 1) to (4132, 5) and from (4231, 2) to (4321, 4). This graph is such that any of its spanning trees satisfies the properties for our T_5. It can be checked that this graph has 420 spanning trees. This gives the lower bound. □

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