de Bruijn Sequences and Universal cycles
SOLUTIONS

A For starters

A.1 **Answer.** No.

Assume that such an arrangement exists. Assume that some number, e.g. number 1, appears k times along the circle. Then it is contained in $5k$ contiguous 5-tuples. On the other hand, this number should appear exactly once in any 5-tuple containing it. The total number of such 5-tuples is $\binom{99}{4}$, so we have $k = \binom{99}{4}/5$, which is not an integer. A contradiction.

A.2 **Answer.** Such a sequence exists.

To prepare, let us enumerate all 5-tuples of positive integers: the first one is $(1, 2, 3, 4, 5)$, then follow all the tuples with the sum 16, and so on.

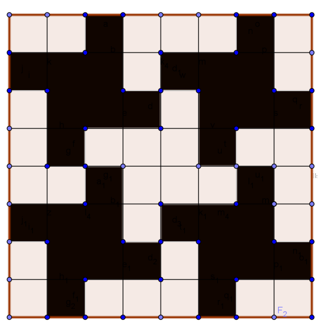
We will construct our sequence iteratively. Assume that at some moment we have obtained a finite sequence such that it contains all 5-tuples from 1st to kth (together with some extra 5-tuples), and no 5-tuple occur twice. We will augment this sequence by some terms so that it will satisfy the same condition with k replaced by $k + 1$.

If the $(k + 1)$st tuple already occurs somewhere, then we do nothing. Otherwise, if $(a_1, a_2, a_3, a_4, a_5)$ is the $(k+1)$st tuple, we will augment the sequence by ten numbers $b_1, b_2, b_3, b_4, b_5, a_1, a_2, a_3, a_4, a_5$, where $b_1, \ldots, b_5$ are distinct numbers which are greater than all used numbers (including the numbers $a_1, \ldots, a_5$). It is easy to see that we have obtained a desired prolongation.

Acting so, after an infinite time we will get an (infinite) sequence containing each 5-tuple exactly once. Indeed, by the construction every 5-tuple occurs in this sequence; if some tuple occurs twice, then this happens on some finite initial segment of our sequence, which is impossible.

A.3 a) **Answer.** Yes.

A picture below shows an 8×8 piece of plane colored in a required manner. One may look out of the window and try to find a similar pattern.



б) **Answer.** No.

We will establish the answer by an easy case distinction.

Arguing indirectly, we have that all “diagonal squares” within one period are colored distinctly. We depict the further arguments on a 4×4 square which is a “period” of our coloring; the two colors are replaced by zeroes and ones, respectively. To start, let us find a square with four zeroes in it.

We claim that all cells marked by stars should contain ones. Due to the symmetry, it suffices to deal with one of these cells. Assume that A contains 0; then both B and C contain ones, otherwise there would appear two

	*		*
*		0	
	0		0
*		0	

squared with four zeroes in each. But then we will have two squares having the pattern “one at the top, zeroes in all other positions”, which is prohibited.

	B		C
A		0	
	0		0
		0	

So all the starred cells contain ones. Thus we already have all eight arrangements with an even number of ones. Notice also that each “diagonal square” contains either four filled cells or four cells which are still empty. Those eight empty cells thus should contain all arrangements with an odd number of ones.

Without loss of generality, four of these cells contain the digits as shown in the picture below.

A		B	
	1		C
0		0	
	0		D

Assume that cell A contains 0. Then the parity reasons yield that the cells B and C contain zeroes, while D contains 1; one can easily see that this arrangement does not fit.

Otherwise A contains one; then B and C contain ones as well, while D contains 0. This arrangement also does not fit.

B de Bruijn sequences and directed graphs

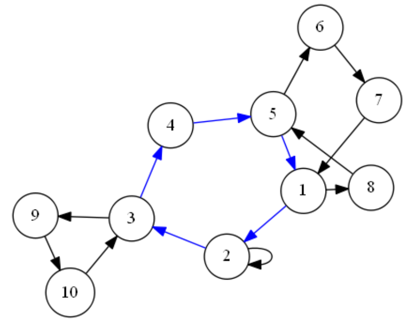
B.4 Assume that an Eulerian tour exists. If we move along this tour, we come into each vertex as many times as we leave it, so the graph must be balanced. Next, while moving along the cycle one can get from any vertex to any other vertex, so the graph is strongly connected.

Conversely, assume that a directed graph is strongly connected and balanced. We will prove that an Eulerian tour exists by induction on the number of edges in this graph. The base case when a graph has no edges is trivial.

For the step, let us start from any vertex and move along the edges until we come to an already visited vertex. Thus we have found some cycle C in our graph.

Now let us delete the edges of this cycle from our graph. The remaining graph is still balanced, but it might become disconnected.

Lemma. *If a balanced graph is weakly connected (i.e. if it is connected when we regard it as a usual non-directed graph), then it is connected.*



Proof. Take an arbitrary vertex v. Let A be a set of vertices reachable from v, and let B be the set of all other vertices. Assume that B is nonempty. Then there are no edges from A to B, but there should be some edges from B to A due to the weak connectedness. Now comparing the total in- and out-degree of the vertices of A we come to a contradiction: these sums should be equal due to balancedness, but the first one is greater due to the edges from B to A. A contradiction. $\square$

Thus, after C has been deleted, the graph splits into several components each of which is balanced and connected, so each has an Eulerian tour. Then a desired tour can be constructed as follows: we move along the edges of C until we cone to some vertex of some component; then we walk along the Eulerian tour of this component, and then we move further along C, and so on.

B.5 Let us construct a special graph which is called a *de Bruijn graph* $G(n,k)$. Its vertices are all k^{n-1} $(n-1)$-letter words in a k-letter alphabet, and a vertex of the form $a_1 \dots a_{n-1}$ is connected to every vertex of the form $a_2 \dots a_n$. Thus the edges correspond to n-letter words of the form $a_1 \dots a_n$, and there are k^n edges.

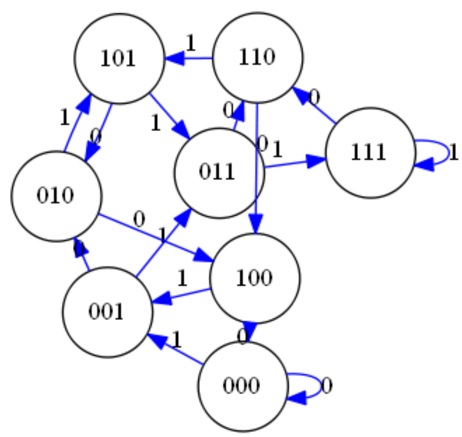


Рис. 1: $G(4,2)$.

More generally, each word of length $n+a$ corresponds to an (oriented) path of length $a+1$ in $G(n,k)$. The vertices of this path are just all $(n-1)$-letter subwords of our word in the order of their appearance.

Let us write on each edge $a_1 \dots a_{n-1} \to a_2 \dots a_n$ the last letter a_n of the corresponding word. Thus, if the last edge of a path corresponds to the word $a_1 \dots a_k$, then the letters written on the previous $k-1$ edges form the word $a_1 \dots a_{k-1}$.

Every in- and out-degree of a vertex in G is equal to k. Moreover, the graph is connected, since for every two vertices there exists a path connecting them (this path corresponds to a concatenation of these words).

Thus graph $G(n,k)$ hass an Eulerian tour. The sequence of letters written on the edges of this cycle form a de Bruijn sequence, since all the subwords correspond to the different edges of our graph.

B.6 We will use induction on k. Base fro $k=0$ is clear.

If we can mark some edges, so that there is precisely one marked edge going from each vertex and precisely one edge coming to each vertex, then we will be able to color marked edges with color 1 and use the inductive assumption for $k-1$.

We will apply *Hall's lemma*: If there are several gentlemen and ladies, so that every m gentlemen are acquainted with at least m ladies in total (for each m), then each gentleman can propose to a lady he knows, so that there will be no conflicts.

Put a gentleman and a lady in each vertex and say that a particular gentleman is acquainted with a particular lady if there is an edge from him to her. Assume that conditions of the Hall's lemma are not satisfied: there is a group of m_1 gentlemen who are acquainted in total with $m_2 < m_1$ ladies. From vertices in which these gentlemen are located a total km_1 edges are emanating, and these edges are all going to a set m_2 vertices. So at least one of them has indegree at least $\frac{km_1}{m_2} > k$, contradiction.

Now we can just select pairs «gentleman-lady» with the help of the Hall's lemma and mark the corresponding edges.

B.7 b) Suppose we are working over the alphabet $\{0, 1, \ldots, k-1\}$. Algorithm will be the following: start with $k-1$ zeroes, on each step put the maximal possible letter, such that no word of length k is repeated. If it is possible to make $k^n - n + 1$ such steps, then the resulting cyclic sequence will be of de Bruijn type.

In other words, given the de Bruijn graph $G(n, k)$, start with the vertex $000\ldots0$ and walk along the edges, each time choosing an edge we haven't used yet with the maximal possible label.

Suppose we are stuck at some vertex. Then it must be $00\ldots0$.

We have already gone through the edge $00\ldots0$, for otherwise we could have proceeded along it.

Consider all unused edges and color those with label 0 in black and those with nonzero label in gray.

Lemma: There is no oriented cycle of black edges.

Assume there exists one. Then if we go along it, at some point we must go through the edge $0\ldots00$, but it is not colored. Contradiction.

Suppose there exists at least one black or gray edge v_1v_2. We came to v_2 less than k times, so we left it less than k times and didn't use the edge with label 0. So there is some black edge v_2v_3, then some black edge v_3v_4 and so on. As a result we get an oriented cycle of black edges which contradicts the lemma.

Part a) is a particular case for $k = 2$.

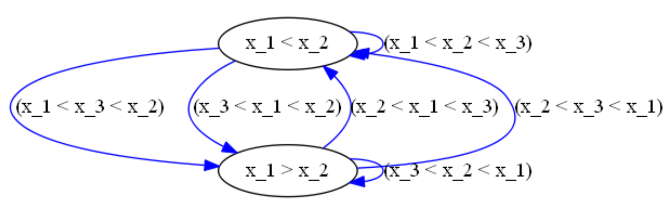
C Magic tricks and universal cycles

C.1 Let $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ be two sequences such that the numbers in each sequence are pairwise distinct. We will say that these sequences are *equiordered* if for every pair of indices $i \neq j$ we have either $a_i < a_j$ and $b_i < b_j$, or $a_j > a_i$ and $b_j > b_i$.

Let S_n be the set of orderings of the set $\{1, 2, \ldots, n\}$. Then S_n contains $n!$ elements, each of which is a *permutation of length* n. The magician will be able to perform the trick if he finds a cyclic sequence of length $n!$ such that no two of $n!$ its pieces of length n are equiordered.

For this purpose, we construct a directed graph $\mathbb{S}_n$ whose vertices are the permutations of length $n-1$. The edges are constructed as follows. For every permutation $s \in S_n$ we define its *head* as the permutation of length $n-1$ equiordered with the first $n-1$ elements of s. A *tail* of s is defined similarly (e.g., the head and the tail of 31542 are 2143 and 1432, respectively). Now for every $s \in S_n$ we draw an edge from its head to its tail.

It is convenient for us to write the elements of S_n as the system of inequalities in variables $x_1, \ldots, x_n$; see example in the picture below.



Graph $\mathbb{S}_3$

First, let us show that $\mathbb{S}_n$ admits an Eulerian tour. Indeed, the out-degree of every vertex is n (this is the number of ways to put a new element onto a line relatively to $n-1$ existing ones), as well as the in-degree of every vertex. To check the connectedness, take any two permutations s_1 and s_2 of length $n-1$ and consider some sequence t of $2n-2$ distinct numbers such that the first $n-1$ of them are equiordered with s_1, while the last ones are equiordered with s_2. Then there exists a path from s_1 to s_2 whose edges are determined by the n-element pieces of t.

Let us now choose any Eulerian tour in $\mathbb{S}_n$; let its edges correspond to the permutations $s_1, s_2, \ldots, s_{n!}$ (in order of appearance in the tour). Take $n!$ variables $y_1, \ldots y_{n!}$ (we assume that $y_{n!+k} = y_k$). Each permutation s_i defines the system of inequalities in variables $x_1, \ldots, x_n$; let us write down this system replacing each x_k by y_{i+k-1}. For instance, if s_3 determines the system $x_2 > x_1 > x_3$, then we write down the inequalities $y_4 > y_3 > y_5$. Our aim is to plug the numbers in all y_i's so that all the written inequalities hold.

For every permutation s_i, let us draw $\frac{n(n-1)}{2}$ oriented edges between $y_i, \ldots, y_{i+n-1}$, namely we draw an arrow $y_{i_1} \to y_{i_2}$ if $y_{i_1} > y_{i_2}$ in s_i. We obtain a new directed graph H on the vertices $y_1, \ldots, y_{n!}$.

Lemma. *Graph H contains no oriented cycles.*

Proof. We say that an edge $y_i \to y_j$ of H *faces right* if $j - i \in \{1, 2, \ldots, n-1\}$; similarly, it *faces left* if $i - j$ belongs to the same set. If $n > 2$ then every edge of H faces exactly one of two directions; moreover, if H contains an edge $y_i \to y_j$ then it does not contain $y_j \to y_i$.

Assume the contrary. Consider an oriented cycle C in H with the smallest number of vertices. Suppose that C contains two consecutive edges facing different directions — say, $y_i \to y_j$ facing left and $y_j \to y_k$ facing right. Both these edges are drawn according to the permutation s_j, so the edge $y_i \to y_k$ also exists by the same reason. Therefore, the cycle is contractible, which contradicts our choice.

Thus, all the edges of C face the same direction — say, all of them face left. Now, there exists an index t such that permutation s_t defines the inequalities $x_1 < x_2 < \cdots < x_n$, which provide the edges $y_{t+n-1} \to y_{t+n-2} \to \cdots \to y_t$ in H. But cycle C should contain an edge connecting two of the vertices $y_t, \ldots, y_{t+n-1}$; this edge then faces right. This contradiction finishes the proof. $\square$

Now, as H contains no oriented cycles, we may introduce a partial order on the variables y_i by setting $y_{i_1} \succ y_{i_2}$ if H contains a path from y_{i_1} to y_{i_2}.

Now we may take any variable which is minimal with respect to this order and assign 1 to it. Remove it from the set of variables, take any minimal variable in the remaining set and assign 2 to it, and so on. Clearly, the assigned values satisfy all the required inequalities.

C.2 The solution can be found in the article J. R. Johnson, Universal cycles for permutations, Discrete Math., 309(2009), pp. 5264–5270.

C.3 The solution is similar to that of **C.1**.

We say that two sequences $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are *equipartitioned* if for every two indices $i \neq j$ we have either $a_i = a_j$ and $b_i = b_j$, or $a_i \neq a_j$ and $b_i \neq b_j$. Denote by P_n the set of all possible partitionings of an n-term sequence.

One may check that $|P_5| = 52$, so the magician will be able to perform the trick if he finds a cyclic sequence of length 52 in which no two 5-tuples of consecutive numbers are equipartitioned. Here is one such sequence:

DDDDDCHHHCCDDCCCHCHCSHHSDSSDSSHSDDCHSSCHSHDHSCHSJCDC

Here the letters S, C, D, and H denote the usual card suits, and a letter J denotes a joker. Thus one may take a usual deck with 52 cards, replace a queen of spades with a joked, and arrange the obtained deck in order to be able to perform this trick.

Now we will show how to minimize a case consideration in finding such a sequence (although the result may involve more than five distinct numbers). We work with P_5, but a similar method may be applied to any P_n with $n > 4$.

Firstly, we construct a directed graph $\mathbb{P}_5$. Its vertices are the elements of P_4, and the edges correspond to the elements of P_5 in a usual manner. Namely, for every $p \in P_5$ we define its *head* and *tail* as two partitions in P_4 equipartitioned with the first and with the last $n-1$ elements of p, respectively. Now for every $p \in P_5$ we draw an edge from its head to its tail. For example, the partition 12324 determines the edge $1232 \to 1213$.

Graph $\mathbb{P}_n$ is connected. Indeed, if we need to find a path from p_1 to p_2, we may consider a $(2n-2)$-term sequence such that its first half is equipartitioned with p_1, and its second half is equipartitioned with p_2. The n-term pieces of this sequence determine the edges of a required path.

Next, this graph is also balanced. If an element of P_4 contains k distinct numbers, then every its incoming and outcoming edge corresponds to the addition of a new element which may be either equal to an existing one or distinct from all of them. Thus both in- and out-degree of this vertex are equal to $k+1$.

Thus graph $\mathbb{P}_5$ admits an Eulerian tour. Every such tour defines the system of equalities and inequalities on the cyclic set of variables $y_1, \ldots, y_{52}$ in a way similar to that described in **C.1**.

Now for every equality $y_i = y_j$ we onnect the variables y_i and y_j by a white edge, while for every inequality $y_i \neq y_j$ we connect y_i with y_j by a black edge. We say that this system is *contradictory* if there exists a white path whose endpoints are connected with a black edge (this means that these endpoints are equal and not equal simultaneously). If the system is not contradictory then one may obtain a required arrangement by assigning equal values to those variables which are connected with a white path.

Thus it remains to show the existence of an Eulerian tour determining a non-contradictory system.

First of all, we may notice (by an analogy with **C.1**) that in a contradictory system always exists a contradiction of the form

$$y_{i_1} = y_{i_2} = \cdots = y_{i_k} \neq y_{i_1},$$

where all edges $y_{i_j} \to y_{j+1}$ face right.

Consider a sequence $W = 113112222213311$. It corresponds to a path with 11 edges in $\mathbb{P}_5$.

Lemma 1. *If an Eulerian tour contains this path then it determines a non-contradictory system.*

Proof. Consider 15 variables involved in our path (without loss of generality, they are $y_1, \ldots, y_{15}$). It is easy to see that every path facing right and connecting one of $y_1, \ldots, y_5$ to one of $y_{11}, \ldots, y_{15}$ contains at least two black edges. Thus there is no contradiction of the abovementioned form. □

Lemma 2. *There is an Eulerian tour in $\mathbb{P}_5$ containing the path determined by W.*

Proof. Delete the edges of our path from $\mathbb{P}_5$ and add one edge from the beginning of this path to its end. It suffices to prove that the obtained graph is balanced and connected. The former property is obvious; for the latter, it suffices to show the weak connectedness.

In other words, we need to prove that every 4-term sequence may be extended to the right in a way that the last 5 digits are distinct, and no its 5-letter subword is equipartitioned with some subword in W.

Consider a sequence (a, b, c, d) (some symbols may be equal). Let us try to augment it by a new letter e distinct from the existing ones; this works if a word (a, b, c, d, e) is not equipartitioned with a subword of W. Then we try to repeat this procedure; itf it works four times then we are done. Thus, it remains to check the cases when this algorithm fails at some stage, i.e. when the resulting word is equipartitioned with some subword W' of W.

The last letter of W' should be distinct from all other its letters. Thus we have only three options for W'.

(a) Assume that $W' = 12113$. Then, instead of extending the word 1211 by 3 we may extend it as 12112345.

(b) The case $W' = 33331$ is impossible since all the edges from the vertex 0000 are already used in W (so this vertex is deleted from the new graph).

(c) Assume that $W' = 33312$. Then, instead of extending the word 3331 by 2 we may extend it as 33313245.

The lemma is proved. □

The two lemmas above provide the desired result.

C.4 Consider a complete graph on n vertices labeled by $1, 2, \ldots, n$. Then we need just to find an Eulerian tour in this (undirected) graph. For known reasons, such a tour exists when n is odd, and it does not exist when n is even.

C.5 Consider a circle partitioned into n equal arcs, and enumerate the partitioning points consecutively from 1 to n. Then a triple of numbers determines the partition of this circle into three arcs. We say that two triples *have the same difference type* if the lengths of arcs in corresponding partitions are the same (possibly after a cyclic shift). (To find the arcs lengths one may just order the numbers and calculate their cyclic differences modulo k.)

Example: Let $n = 8$. Then the triples $\{1, 3, 7\}$ and $\{1, 5, 3\}$ have the same difference type $(2, 2, 4)$, while the triples $\{1, 2, 5\}$ and $\{1, 4, 5\}$ have distinct difference types $(1, 3, 4)$ and $(3, 1, 4)$, respectively.

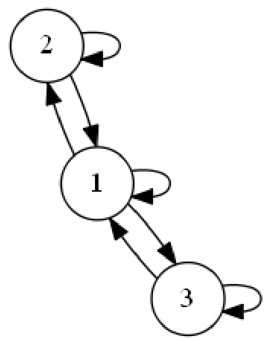
If n is not divisible by 3, then the number of difference types is $\binom{n}{3}/n = \frac{(n-1)(n-2)}{6}$.

To start, let us construct a desired arrangement for $n = 8$. There are 7 difference types:

$$(1, 1, 6), \quad (2, 2, 4), \quad (2, 3, 3), \quad (1, 2, 5), \quad (5, 2, 1), \quad (1, 3, 4), \quad (4, 3, 1).$$

We want to choose in each difference type (x, y, z) one of ordered pairs $x \to y$, $y \to z$, or $z \to x$. In the types above we make it as follows:

$$1 \to 1, \qquad 2 \to 2, \qquad 3 \to 3, \qquad 1 \to 2, \qquad 2 \to 1, \qquad 1 \to 3, \qquad 3 \to 1.$$



These arrows form a balanced connected directed graph. This graph admits an Eulerian cycle, and the sum of the numbers in its period is congruent to 5 modulo 8:

$$(1 \to 1 \to 2 \to 2 \to 1 \to 3 \to 3 \to)1 \to \dots$$

Extending this cycle periodically we get an infinite sequence (a_k). Define the sequence (b_k) as $b_k = a_1 + \cdots + a_k \mod 8$; this sequence is periodic with period length $7 \cdot 8 = 56$.

Let us show that every triple appears in this sequence. Indeed, every triple may be represented as $(x, x+k_1, x+k_1+k_2) \mod 8$ in a way that the numbers k_1, k_2 appear consecutively in (a_i). Let $a_i = k_1, a_{i+1} = k_2$. Then we have $a_{i+8n} = k_1, a_{i+1+8n} = k_2$ for all n. Since $a_1 + \cdots + a_8 \equiv 5 \mod 8$, there exists n such that the number $b_{8n+i-1} \equiv a_1 + \cdots + a_{i-1} + n(a_1 + \cdots + a_8)$ is congruent to x modulo 8. Thus the terms $b_{8n+i-1}, b_{8n+i}, b_{8n+i+1}$ form a required triple.

Now we pass to a general construction.

Lemma. *For every $n \geq 8$ not divisible by 3 one may choose an ordered pair in each difference type so that the $\frac{(n-1)(n-2)}{6}$ obtained oriented edges form a connected balanced directed graph. (Naturally, the vertices of this graph are exactly those numbers which have at least on outcoming edge.)*

Proof. Consider two cases separately.

Case 1: n is even. If a difference type contains two equal numbers, we connect them with a loop. Thus we obtain loops $i \to i$ for all i from 1 to $n/2 - 1$. If all three numbers are distinct then we connect two smallest ones. The obtained graph on vertices $1, \dots, n/2 - 1$ is balanced, since the edges $i \to j$ and $j \to i$ appear simultaneously. It is connected since each of the numbers $2, \dots, n/2 - 1$ is connected with 1 in both directions.

Case 2: n is odd. The set of vertices is $\{1, 2, \dots, (n-1)/2\}$. If a type contains two equal numbers, then we connect them with a loop. Otherwise we connect two smallest numbers, except for two cases. The two exceptional triples are $(2, (n-1)/2 - 1, (n-1)/2)$ и $(2, (n-1)/2, (n-1)/2 - 1)$, in which we choose the edges $(n-1)/2 \to 2$ and $2 \to (n-1)/2$, respectively.

The obtained graph is again balanced. Its connectedness follows from the fact that every number except for $(n-1)/2$ is connected with 1, and $(n-1)/2$ is connected with 2. $\square$

Denote the constructed graph by L_n; it has $[\frac{n-1}{2}]$ vertices, and it contains a loop $1 \to 1$. We need also a larger graph $\mathbb{L}_n$. It contains $n[\frac{n-1}{2}]$ vertices, each having the form (k, i) where $k \in \{1, 2, \dots, n\}$, and i is a vertex of L_n. For every edge $i \to j$ in L_n we introduce n edges in $\mathbb{L}_n$; these edges have the form $(k, i) \to (k+i, j)$ for all admissible k.

The graph $\mathbb{L}_n$ is balanced since L_n is such. Let us show that $\mathbb{L}_n$ is connected. Due to balancedness, it suffices to show a weak connectedness, so we will just show that there exists a path from any vertex (k, i) to $(0, 1)$.

In L_n, the vertices i and 1 are connected with a path; this path induces a path in $\mathbb{L}_n$ from (k, i) to some vertex of the form $(s, 1)$. But $\mathbb{L}_n$ also contains all edges of the form $(t, 1) \to (t+1, 1)$, and one can move along such edges to reach $(0, 1)$ from $(s, 1)$.

Thus $\mathbb{L}_n$ admits an Eulerian tour. Its length is $\binom{n}{3}$, and for every k and every edge $i \to j$ of L_n this cycle contains three consecutive vertices of the form $(k, i), (k+i, j), (k+i+j, *)$ (we do not care about the starred number). Thus, if we write down the sequence of the first coordinates of the vertices in this tour we get the sequence containing every triple of numbers from 1 to n.

D The beginning of enumeration

D.1 A graph $\mathcal{L}G$ is strongly connected precisely when for every two edges e_1, e_2 in G there exists a (directed) path starting from e_1 and arriving at e_2.

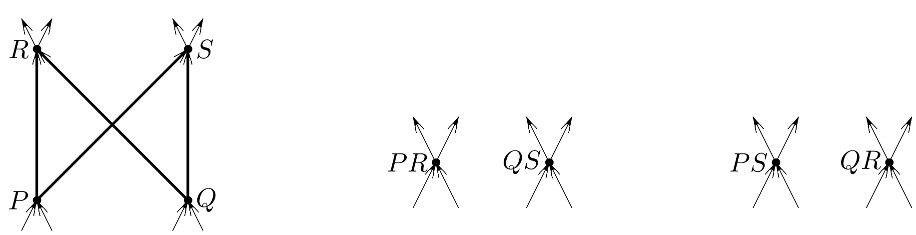
Assume that G is stronly connected, and e_1 and e_2 are two its edges. There exists a path T from the target of e_1 to the source of e_1; then $e_1 T e_2$ is a desired path in $\mathcal{L}G$.

Assume now that $\mathcal{L}G$ is strongly connected. If every vertex of G is a source of some edge as well as a target of some edge, then G is connected. Indeed, to find a path from v_1 to v_2 it suffices to find a path in $\mathcal{L}G$ from some edge with v_1 as a source to some edge with v_2 as a target.

So, assume that this condition fails, e.g., there is no edge from some vertex v. Since v is not isolated, there exists an edge e going to v, and there exists some other edge e' in our graph. But then it is impossible to reach e' from e in $\mathcal{L}G$. Contradiction.

D.2 The ingoing and outgoing degrees of each vertex in $\mathcal{L}G$ are also equal to 2. It has vertices P, Q, R, S, and there are edges from P and Q to R and S.

$\mathcal{L}G_1$ can be obtained from $\mathcal{L}G$ in the following fashion: edges PS, PR, QS и QR are deleted, P and R are glued into a vertex (PR), Q and S are glued into a vertex (QS). $\mathcal{L}G_2$ is similar: the same four edges are deleted, vertices P and S are glued into (PS), vertices Q and R are glued into (QR).



Graph $\mathcal{L}G$ Graph $\mathcal{L}G_1$ Graph $\mathcal{L}G_2$

We will call edges PR, QS, PS and QP of $\mathcal{L}G$ the *special* ones, as well as vertices (PR), (QS), (PS) and (QP) in $\mathcal{L}G_1$ and $\mathcal{L}G_2$.

By a *great 4-tuple* we mean an unordered set of four paths in $\mathcal{L}G$ such that their edge sets are disjoint and cover all edges of $\mathcal{L}G$ but the special ones. So each great 4-tuple contains two paths starting at R, two paths starting at S, two paths ending at Q, two paths ending at P.

A great 4-tuple corresponds to a 4-tuple of disjoint paths in $\mathcal{L}G_1$ ($\mathcal{L}G_2$), covering all edges, starting and ending at special vertices, but not going through them in the mean time. For simplicity we will say that a great 4-tuple is the same in $\mathcal{L}G$, $\mathcal{L}G_1$ and $\mathcal{L}G_2$.

Each Eulerian cycle in $\mathcal{L}G$ is split by the special edges into four pieces producing a great 4-tuple. In a similar fashion, each Eulerian cycle in $\mathcal{L}G_1$ or $\mathcal{L}G_2$ is split into a great 4-tuple by occurences of the special vertices.

To solve the problem it suffices to show that each great 4-tuple corresponds to four Eulerian cycles in $\mathcal{L}G$, while to two Eulerian cycles in $\mathcal{L}G_1$ or $\mathcal{L}G_2$.

A great 4-tuple in $\mathcal{L}G$ can be of one of three types:

(a) One path $R \to P$, one path $R \to Q$, one path $S \to P$, one path $S \to Q$.
(b) Two paths $R \to P$ and two paths $S \to Q$.
(c) Two paths $R \to Q$ and two paths $S \to P$.

Let us go through all cases.

(a) Denote the paths of a great 4-tuple by $R \dots P$, $R \dots Q$, $S \dots P$ и $S \dots Q$. There are precisely 6 ways to put on them a cyclic order. Each can be written as

$$X_1 \dots Y_1 X_2 \dots Y_2 X_3 \dots Y_3 X_4 \dots Y_4,$$

where X_i are the starting points of the paths and Y_i the endpoints. We get four pairs of vertices $Y_i X_{i+1}$. If these pairs coincide with $\{PS, PR, QS, QR\}$, then we get an Eulerian cycle in $\mathcal{L}G$. If they coincide with $\{PR, PR, QS, QS\}$, then we get an Eulerian cycle in $\mathcal{L}G_1$, and if they coincide with $\{PS, PS, QR, QR\}$ we get an Eulerian cycle in $\mathcal{L}G_2$. We will write down these ways.

(i) $R \dots PR \dots QS \dots PS \dots Q$ — set $\{PR, QS, QR, PS\}$, a cycle in $\mathcal{L}G$;
(ii) $R \dots PR \dots QS \dots QS \dots P$ — set $\{PR, QS, QS, PR\}$, a cycle in $\mathcal{L}G_1$;
(iii) $R \dots PS \dots PR \dots QS \dots Q$ — a cycle in $\mathcal{L}G$;
(iv) $R \dots PS \dots PS \dots QR \dots Q$ — a cycle in $\mathcal{L}G_2$;
(v) $R \dots PS \dots QR \dots QS \dots P$ — a cycle in $\mathcal{L}G$;
(vi) $R \dots PS \dots QS \dots PR \dots Q$ — a cycle in $\mathcal{L}G$.

So we get four cycles in $\mathcal{L}G$, one cycle in $\mathcal{L}G_1$ and one cycle in $\mathcal{L}G_2$.

(b) It will be similar, but two pairs of different paths $R \dots P$ and $S \dots Q$ will be denoted in the same way.

(i) $R \dots PR \dots PS \dots QS \dots Q$ — a cycle in $\mathcal{L}G$;
(ii) $R \dots PR \dots PS \dots QS \dots Q$ — a cycle in $\mathcal{L}G$. One mustn't forget that this is a different order, since two paths of the great 4-tuple are switched
(iii) $R \dots PS \dots QR \dots PS \dots Q$ — a cycle in $\mathcal{L}G_2$;
(iv) $R \dots PS \dots QS \dots QR \dots P$ — a cycle in $\mathcal{L}G$;
(v) $R \dots PS \dots QR \dots PS \dots Q$ — a cycle in $\mathcal{L}G_2$;
(vi) $R \dots PS \dots QS \dots QR \dots P$ — a cycle in $\mathcal{L}G$.

So we get four cycles in $\mathcal{L}G$ and two cycles in $\mathcal{L}G_2$.

(c) Same argument as in the previous case.

D.3 Answer: $2^{2^{n-1}-n}$,

Clearly $B(2,n) = \epsilon(G(2,n))$. We will use induction to show that $\epsilon(G(2,n)) = \frac{2^{2^{n-1}}}{2^n}$.

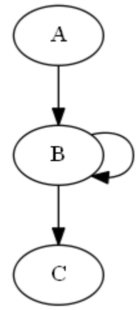
Observe that $\mathcal{L}G(2,n) = G(2,n+1)$. Indeed, vertices of $G(2,n+1)$ correspond to edges of $G(2,n)$, while edges of $G(2,n+1)$ correspond to binary words of length $n+1$, i.e. paths in $G(2,n)$ of length 2.

Lemma. *Suppose a digraph G has n vertices, each of indegree and outdegree 2. Then $\epsilon(\mathcal{L}G) = 2^{n-1}\epsilon(G)$.*

The inductive step immediately follows from the lemma: there are 2^{n-1} vertices in $G(2,n)$, so

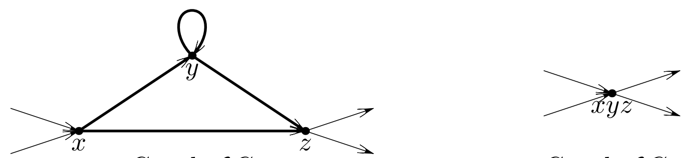
$$\epsilon(\mathcal{L}G(2,n)) = 2^{2^{n-1}-1} 2^{2^{n-1}-n} = \frac{2^{2^n}}{2^{n+1}}.$$

Proof of the Lemma. We will prove it by induction on the number of vertices. Base: $n = 1$. A graph is a vertex with two loops, it has one Eulerian cycle, precisely as its line graph.



Step $n \to n+1$. If G has a vertex with two loops, it is not connected to anything else, so both G and $\mathcal{L}G$ have no Eulerian cycles.

Suppose G has a vertex B with a single loop. Then in a Eulerian cycle the loop $B \to B$ must come in between $A \to B$ and $B \to C$. Erase vertex B, put in edge AC and denote the resulting graph by G_1. $\epsilon(G) = \epsilon(G_1)$. The local difference between $\mathcal{L}G$ and $\mathcal{L}G_1$ is shown on the picture below ($x = A \to B, y = B \to B, z = B \to C$).



Graph $\mathcal{L}G$ Graph $\mathcal{L}G_1$

Each Eulerian cycle in $\mathcal{L}G_1$ corresponds to two cycles in $\mathcal{L}G$: we go twice thorugh the vertex xyz and there are two ways to do it in $\mathcal{L}G$: $x \to y \circlearrowleft y \to z$ and $x \to z$. We can apply them in any order, so we get two cycles. Then

$$\epsilon(G) = \epsilon(G_1) = \frac{1}{2^{n-1}}\epsilon(\mathcal{L}G_1) = \frac{1}{2^n}\epsilon(\mathcal{L}G).$$

Suppose now that we are considering a vertex with no loops in it. Construct G_1 and G_2 as in **D2**. Each of them has $n-1$ vertices and satisfies lemma conditions.. Each Eulerian cycle in G corresponds to a cycle in G_1 or in G_2. As a result, $\epsilon(G) = \epsilon(G_1) + \epsilon(G_2)$,

$$\epsilon(\mathcal{L}G) = 2(\epsilon(\mathcal{L}G_1) + \epsilon(\mathcal{L}G_2)) = 2 \cdot 2^{n-1}(\epsilon((G_1) + \epsilon(G_2))) = 2^n \epsilon(G).$$

Lemma is proved.

E de Bruijn tori

General remark. Throughout this section, we regard letters of the alphabet $X_k = \{1, 2, \ldots, k\}$ as residues modulo k. So it is possible to perform arithmetic operations with these letters and also obtain a letter. We will usually denote the jth cell in the ith row of a square grid by (i, j).

E.1 For convenience, we will construct a torus of type $(k, k^{u-1}, 1, u)_k$; a required torus can be obtained via reflection across the diagonal.

Firstly, assume that $u \geq 3$. Let $c_1, c_2, c_3, \ldots, c_{k^{u-1}}$ be a de Bruijn sequence of rank $u-1$ in the alphabet $X_k = \{1, 2, \ldots, k\}$; we extend it to obtain an infinite k^{u-1}-periodic sequence. Now let us fill the cell (i, ℓ), $\ell > 0$ in a square grid with the number

$$a_{i,\ell} = i + \sum_{s=1}^{\ell} c_s$$

(recall that addition is performed modulo k). Similarly extend it to the filling of the whole square grid.

Each number appears in the period $c_1, c_2, \ldots, c_{k^{u-1}}$ exactly k^{u-2} times; since $u \geq 3$, the sum $\sum_{s=1}^{k^{u-1}} c_s$ is divisible by k. So, the arrangement is indeed k^{u-1}-periodical horizontally and k-periodical vertically, thus it is a torus of required size. (One may notice that the periodicity property also holds if $u = 2$ and k is odd.)

It remains to prove that that every u numbers $x_0, \dots, x_{u-1}$ appear (in this order) consecutively in some row. Since (c_i) is a de Bruijn sequence, there exists an index ℓ such that $c_{\ell+i} = x_i - x_{i-1}$ for all $i = 1, \dots, u-1$. Since the numbers $a_{1,\ell}, \dots, a_{k,\ell}$ are pairwise distinct, there exists an index r such that $a_{r,\ell} = x_0$. Then by our construction we have

$$a_{r,\ell+i} = a_{r,\ell} + \sum_{s=1}^{i} c_{\ell+s} = x_0 + \sum_{s=1}^{i} (x_s - x_{s-1}) = x_i,$$

so we have found a desired $1 \times u$ rectangle.

Now assume that $u = 2$ and $k \geq 3$. We need to fill the rows of a $k \times k$ torus by the numbers $1, 2, \dots, k$ so that every pair of numbers appears as a pair of consecutive numbers in some row. In terms of a de Bruijn graph $G(2, k)$ introduced in the solution of **B.5**, we need to split the edges of this graph into k cyclic tours, each of length k.

If k is odd, say $k = 2t + 1$, then one may use the tours

$$\big(a \to (a+1) \to a \to (a+2) \to a \to \cdots \to a \to (a+t) \to a \to a\big)$$

for $a = 1, 2, \dots, k$. (One may also mention that the model for $u \geq 3$ works in this case as well.)

If k is even, say $k = 2t$, then we may start with the tours

$$C_a = \big(a \to (a+1) \to a \to (a+2) \to a \to \cdots \to a \to (a+t-1) \to a\big)$$

for all $a = 1, 2, \dots, 2t$. Each of these tours has length $2t-2$, and they cover all the edges except for those of the forms $x \to x$ and $x \to (x+t)$. Next, we augment each tour C_{2i-1} by two edges $(t+i-1) \to (i-1) \to (t+i-1)$, and we augment each tour C_{2i} by two loops $2i \to 2i$ and $(2i+1) \to (2i+1)$. One can easily check that this is possible, because these tours contain the vertices which are needed for such an augmentation.

E.2 Let $\dots, c_1, c_2, \dots, c_{k^2}, \dots$ be a de Bruijn sequence of rank 2 in the alphabet X_k (extended periodically). We put this sequence into each row with different shifts. Namely, in the ith row we shift this sequence to the right by $0 + 1 + \cdots + (i-1)$ cells. Thus, the sequence in the ith row is shifted by $i-1$ with respect to the $(i-1)$th row. Moreover, the sequence in the (k^2+1)th column is shifted (in total) by $1 + 2 + \cdots + k^2$; this last sum is divisible by k^2, thus our arrangement indeed defines a $k^2 \times k^2$ torus.

Now, for every 2×2 square, its first and second rows appear in our de Bruijn sequence. Assume that the occurrence of the second one is s terms to the right relatively to the occurrence of the first one. Then this square appears in the union of the sth and the $(s+1)$th rows.

E.3 One may modify the method from the previous solution in order to work here as well. Again, we will construct a torus of type $(k^{u(v-1)}, k^u, v, u)_k$. For convenience, denote $R = k^{u(v-1)}$ and $S = k^u$.

We start with the de Bruijn sequence of rank u (its period length is S). We put it into the ith row shifting it to the right with respect to the previous row by some number d_i which will be defined later. (So the ith column is shifted by $d_1 + \cdots + d_i$ relatively to the zeroth one.) Surely we should have $d_{i+R} = d_i$.

In order to provide a vertically R-periodic arrangement, the sum $d_1 + \cdots + d_R$ should be divisible by S. Next, as in the previous problem, in order to obtain all possible $v \times u$ rectangles, we just need all the combinations of $v-1$ shifts to appear in the sequence (d_i). Thus (d_i) should be just a de Bruijn sequence of rank $v-1$ in the alphabet X_S.

In particular, this means that the sum of the elements in the period of (d_i) is just

$$k^{u(v-2)} \cdot (1 + 2 + \cdots + k^u),$$

and this number is indeed divisible by k^u for all $k, u, v \geq 2$, except for the case when $v - 2 = 0$ and k^u is even.

E.4 $k = 2$ was shown in problem **A.3a)**; thus we will assume that $k \geq 4$.

Step 1. Let us construct a rank 2 de Bruijn sequence $C = (c_1, \dots, c_{k^2})$ with an additional property that $c_{k^2/2} = c_{k^2}$. Consider a de Bruijn graph $G(2, k)$ and split all its edges into groups of the form $M_i = \{(\ell, \ell+i) : \ell = 1, 2, \dots, k\}$. Then let us merge all these groups into two equipotent sets so that M_1 and M_{k-1} are in distinct sets. Then we have split all the edges into two subgraphs, and both are connected and balanced.

Let us take some Eulerian tours in these subgraphs and glue them at a common vertex v. We get the Eulerian tour in the whole graph, and two of the occurrences of v are half a tour apart from each other. This provides (by the methods used in **B.5**) exactly a required de Bruijn sequence.

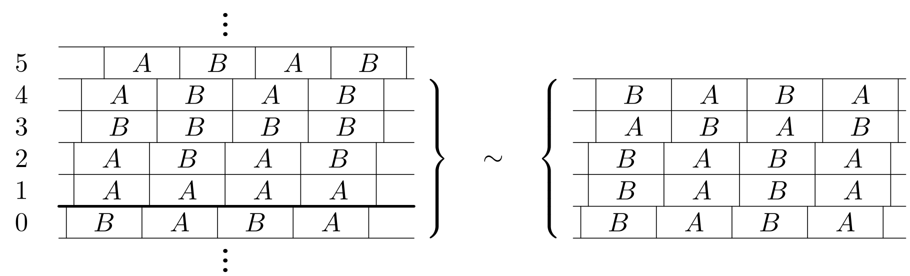
Step 2. Now we perform almost the same procedure as in **E.2**. Consider a constructed de Bruin sequence C; let A and B be its first and second halves. Notice that each sequence of two letters appears either in the sequence with period A or in the sequence with period B. (Here one needs to use the additional property of constructed sequence — why?)

Let us put the sequence with period C into the rows numbered from 4 to k^2 so that the relative shift of two neighboring rows takes all values from 0 to k^2-1, except for 0, 1, $k^2/2$, and $k^2/2+1$. The sum of all the numbers from 0 to k^2-1 is congruent to $k^2/2$ modulo k^2. Then the k^2th (as well as the zeroth) row is shifted relatively to the fourth one by $k^2/2-2$.

Next, we put the same sequence into the second row shifting it by one to the left relatively to the fourth row.

Finally, we put the sequence with period A into the first row, and the sequence with period B to the third one. We shift them so that the parts A in the first and the second rows are put on the same level, and the same holds for the parts B in the third and the fourth rows (see the left picture below).

Due to the changes made in the procedure from **E.2**, we need to check that the 2×2 squares in rows from zeroth to the fourth are the same as they would be if these rows were just the usual de Bruijn sequences with the relative shifts of consecutive rows equal to 0, 1, $k^2/2$, and $k^2/2+1$. It is easy to see that the squares in the rows 1–2 and 3–4 are the same as they would be in two pairs of usual "de Bruijn rows" with relative shifts 0 and $k^2/2$. Similarly, the squares in the rows 0–1 and 2–3 are the same as they would be in two pairs of usual rows with relative shifts 1 and $k^2/2+1$.



E.5 There are several such construction. We will present one using the construction from problem **E.6**.

Assume that $k>2$. Take a de Bruijn torus of type $(k^2,k^2,2,2)_k$ constructed in **E.2** (for odd k) or in **E.4** (for even k). One can check that in both constructions, the sum of letters in each *row* is 0 (it is almost trivial for the usual de Bruijn sequence; for the halves of the sequence used in **E.4**, one needs to involve their explicit construction). Thus one may apply **E.6** to this torus and reflect across the diagonal to get a de Bruijn torus of type $(k^4,k^2,2,3)_k$.

Pitifully, this method does not work for $k=2$, since in the (essentially unique) construction of a de Bruijn torus of type $(4,4,2,2)_2$ the row sums are odd. Here one may apply the method similar to those in the previous problems. Notice that the two columns of the $(4,4,2,2)_2$-torus contain all eight possible 3×1 arrangements; thus one needs only to combine them so that all possible pairings are present. One such arrangement is shown in the picture below.

1	1	0	0	1	0	1	1	1	0	0	1	1	0	1	1
0	0	1	1	0	1	1	1	0	0	1	1	0	0	1	0
0	0	1	1	0	0	0	1	0	1	1	0	0	0	0	1
0	0	1	1	0	0	1	0	0	0	1	1	0	1	1	1

E.6 For further purposes, we will prove a more general statement, where R and S are not necessarily powers of k, but are just positive integers with $RS=k^{uv}$. We act similarly to **E.1** using some additional construction.

Step 1. Now we need a sequence longer than a usual de Bruijn sequence. Namely, we need a cyclic sequence of letters in X_k with period $p_1,p_2,\dots,p_{Sk^v}$ such that for every word $x_1\dots x_v$ and for every number $\ell=1,2,\dots,S$

there exists an index $i \equiv \ell \pmod S$ such that $p_{i+1} = x_1$, $p_{i+2} = x_{i+2}$, $\ldots$, $p_{i+v} = x_v$. So, each word of length v in the alphabet X_k would appear exactly S times in the period.

To reach the goal, we consider a different graph G'. Its vertices have the form $(w; i)$, where w is a word of length $v-1$ in the alphabet X_k, and $i \in \{1, 2, \ldots, S\}$ (the indices i will be regarded as residues modulo S). If the the usual de Bruijn graph $G(v, k)$ contains an edge from w_1 to w_2, then we draw an edge from $(w_1; i)$ to $(w_2; i+1)$ for every $1, 2, \ldots, S$. Clearly, G' is balanced and strongly connected, so there exists an Eulerian tour in this graph. If we now reconstruct a word from this tour in a usual manner, we get the desired sequence. Indeed, the required subword $x_1 \ldots x_v)$ in the position congruent to i modulo S corresponds exactly to the edge $(x_1 \ldots x_{v-1}; i) \to (x_2 \ldots x_v; i+1)$.

Step 2. Now we are ready to construct a desired new torus. Let $a_{i,j}$ denote the letter in position (i, j) of the given torus. Consider a cell (I, J) in the new torus and let $j = J \mod S$. Then we fill this cell with the number

$$A_{I,J} = p_J + \sum_{m=1}^{I} a_{m,j}.$$

By the conditions on the old torus, the new arrangement is vertically R-periodic. Obviously, this arrangement is also horizontally Sk^v-periodic.

Next, each row of this new torus satisfies the same property as the sequence constructed in Step 1. Indeed, the subwords of (p_j) in the positions congruent to i modulo S are just all possible words, and similar subwords of a certain row are obtained from them by componentwise addition of a fixed word.

Now it is easy to see that a constructed torus contains every possible $(u+1) \times v$ rectangle. Let $(x_{i,j})$ be the arrangement of numbers in such a rectangle. Denote $y_{i,j} = x_{i+1,j} - x_{i,j}$. The arrangement $(y_{i,j})$ appears somewhere in the given torus, say in the positions $(\alpha+1, \beta+1)$ to $(\alpha+u, \beta+v)$. Then the new torus contains a desired arrangement in one of k^v positions of the form $(\alpha+1, \beta+1+\mu S)$–$(\alpha+u+1, \beta+v+\mu S)$ with $\mu = 0, 1, \ldots, k^v - 1$, by the same reasoning as in **E.1**.

Remark. Due to the properties of all the rows, it is easily seen that the row sums of the obtained torus are all divisible by k. Thus one may apply the same procedure iteratively, switching the directions every time.

Moreover, one may see that it suffices to switch the direction every second time, provided that Sk^{v-2}. Indeed, let $p_1, \ldots, p_{Sk^v}$ be a period of some row in a constructed torus. The row of the second iteration will have the form

$$\alpha, \quad \alpha + p_1, \quad \alpha + (p_1 + p_2), \quad \ldots, \quad \alpha + (p_1 + \cdots + p_{Sk^v - 1}).$$

The sum of all these elements is congruent modulo k to

$$-p_1 - 2p_2 - \cdots - (Sk^v - 1)p_{Sk^v-1} \equiv -\big(p_1 + p_{k+1} + \cdots + p_{Sk^v-k+1}\big) - 2\big(p_2 + p_{k+2} + \cdots + p_{Sk^v-k+2}\big) - \cdots.$$

In each bracket, every letter appears Sk^{v-2} times; this number is divisible by k, so the total sum is divisible by k as well.

E.7 Starting from the construction of **E.5**, we apply **E.6** thrice obtaining

$$(k^4, k^2, 2, 3)_k \quad \xrightarrow{(1)} \quad (k^4, k^5, 3, 3)_k \quad \xrightarrow{(2)} \quad (k^4, k^8, 4, 3)_k \quad \xrightarrow{(3)} \quad (k^8, k^8, 4, 4)_k;$$

surely sometimes we switch the horizontal and vertical directions. In order to show that **E.6** is applicable in each case, we need to show that the corresponding row/column sums are divisible by k in each of three steps. For step (3) it follows from the Remark after **E.6**. If $k > 2$, then the same holds for steps (1) and (2), as we have obtained the torus of type $(k^4, k^2, 2, 3)_k$ by the same procedure. In the exceptional case $k = 2$, one may check the desired condition manually (on the picture at the end of **E.5**).

It would be better to mention here that not all de Bruijn tori of type $(k^4, k^2, 2, 3)_k$ satisfy the required condition.

E.8 Firstly, let us prove the claim for even $n = 2t$ by the induction on t. The base cases $t = 1, 2$ have already been considered above.

Assume now that the torus of type $(k^{2t^2}, k^{2t^2}, 2t, 2t)$ has already been constructed, and **E.6** was applied last time to increase the vertical side of a torus. Then it suffices to apply **E.6** as

$$\begin{array}{rcl} \left(k^{2t^2}, k^{2t^2}, 2t, 2t\right)_k & \longrightarrow & \left(k^{2t^2}, k^{2t(t+1)}, 2t+1, 2t\right)_k \\ & \longrightarrow & \left(k^{2t^2+2t+1}, k^{2t(t+1)}, 2t+1, 2t+1\right)_k \\ & \longrightarrow & \left(k^{2(t+1)^2}, k^{2t(t+1)}, 2t+1, 2t+2\right)_k \\ & \longrightarrow & \left(k^{2(t+1)^2}, k^{2(t+1)^2}, 2t+2, 2t+2\right)_k. \end{array}$$

Each time, the claim of **E.6** is applicable due to the Remark after its proof.

Assume now that n is odd but k is a perfect square, so $n = 2t+1$ and $k = a^2$. Then we may start a similar induction. For the base case $t = 0$, it suffices to arrange the numbers $1, 2, \dots, a^2$ in the cells of a $a \times a$ square so that the sum in each column is divisible by a^2 (we need this condition in order to be able to apply **E.6**). For that, it suffices to split these numbers into a groups with equal sums, which is known to be possible.

For the induction step, assume that the torus of type $(a^{(2t+1)^2}, a^{(2t+1)^2}, 2t+1, 2t+1)_k$ has already been constructed, and **E.6** was applied last time to increase the vertical side of a torus (if it was applied at all). Then it suffices to apply **E.6** as

$$\begin{array}{rcl} \left(a^{(2t+1)^2}, a^{(2t+1)^2}, 2t+1, 2t+1\right)_k & \longrightarrow & \left(a^{(2t+1)^2}, a^{(2t+1)(2t+3)}, 2t+2, 2t+1\right)_k \\ & \longrightarrow & \left(a^{2t^2+8t+5}, a^{(2t+1)(2t+3)}, 2t+2, 2t+2\right)_k \\ & \longrightarrow & \left(a^{(2t+3)^2}, a^{(2t+1)(2t+3)}, 2t+2, 2t+3\right)_k \\ & \longrightarrow & \left(a^{(2t+3)^2}, a^{(2t+3)^2}, 2t+3, 2t+3\right)_k. \end{array}$$

Again, the applicability of **E.6** is due to the Remark after its proof.

E.9 The case of even n is completely analogous to such case in the previous problem.

Assume now that $n = 2t+1$ and $k = a^2$. If $n = 1$ then the statement is trivial. For $n \geq 3$, we also use the induction on t, but we need to establish the base case $t = 1$. We will obtain it with a help of **E.10**. By this problem, there exists a de Bruijn torus of type $(a^5, a^3, 2, 2)$. Moreover, the analysis of the construction in **E.10** shows that the sum of numbers in each row is divisible by k. Thus we may apply **E.6** twice to obtain

$$(a^5, a^3, 2, 2)_k \quad \longrightarrow \quad (a^9, a^3, 2, 3)_k \quad \longrightarrow \quad (a^9, a^9, 3, 3)_k,$$

as required.

The step of the induction goes exactly as in the previous problem.

E.10 As usual, we denote $k = 2st$, $R = 4st^2$, and $S = 4s^3t^2$.

Step 1. Let us find s cyclic sequences $L_1, L_2, \dots, L_s$, each of length $R = k^2/s$, such that every 2-letter word in alphabet X_k appears as a subword in exactly one of these sequences. This can be done exactly as in Step 1 of **E.4**, but now we need to split all the edges into s groups $G_1, \dots, G_s$, each with R edges, so that these groups define balanced connected subgraphs. The Eulerian tours in these subgraph will then provide the desired sequences.

We split again all the edges of $G(2, k)$ into k groups $M_i = \{(\ell, \ell+i) : \ell = 1, 2, \dots, k\}$. Firstly, we put into G_i the edges of groups M_{i-1} and M_{k-i}; one can see that this already ensures that the resulting subgraphs will be connected. Then we distribute the other groups evenly between G_i's. The construction is finished.

Step 2. Now we will construct a desired torus. Each column will contain just one of the sequences $L_1, \dots, L_s$ shifted somehow. We fix a starting element of each of the sequences L_i; after that, we may speak on the shifts of any of them relatively to any other.

Let $C = (c_1, \dots, c_{s^2})$ be a de Bruijn sequence of rank 2 in the alphabet X_s. Now we fill the columns as follows. Let I be the number of a column, and i be its residue modulo s^2, so $I = i + s^2 \cdot j$. Then this column will contain the sequence L_{c_i}, and it will be shifted by j relatively to the previous column.

Consider the columns from zeroth to Sth. We claim that every 2×2 square appears in these columns. Indeed, for every two indices i, i' the sequence L_i will follow $L_{i'}$ exactly R times, and all their relative shifts will be distinct; this yields our claim.

It remains to find the period of our arrangement. Since $S = Rs^2$, the Sth column will be shifted relatively to the zeroth one by $(0 + 1 + \cdots + (R-1)) \cdot s^2$. If s is even, then this number is divisible by R, so we have obtained a required torus with periods R and S.

Assume now that s is odd. Then, pitifully, the total shift is divisible only by $R/2$. But then we may perform the same change as in **E.4**. Namely, one may see that in Step 1 we can split one of the groups G_i (say, G_2) into two balanced connected halves. Then the corresponding sequence L_2 can be also constructed in order to consist of two parts ending by the same letter. Then the same trick as in **E.4** is applicable. The details are left to the reader.

Remark: There is also a different construction for this problem.

F The continuation of Enumeration

F.1 Let us fix some edge from v; we will assume that it is the first edge in each Eulerian tour. Consider any Eulerian cycle C. For each vertex $u \neq v$, let us mark the last edge from u in tour C. Denote the subgraph defined by the marked edges by T.

Assume that T contains an (unoriented) cycle ω. The out-degree of every vertex in T does not exceed 1, so ω is in fact an oriented cycle. Let e be the edge of ω which appears in C later than the others; let $u = t(e)$. Notice that $u \neq v$, since the out-degree of v in T is 0. Then the edge f of ω starting at u has to occur in C later than e; otherwise, after having passed e in cycle C we would not be able to leave this vertex. This contradiction shows that T is acyclic. Therefore, if we start walking from an arbitrary vertex of G along the edges of T, we will eventually reach a vertex with no outgoing edges; such a vertex should be v. Thus, T is an oriented spanning tree rooted at v.

Thus, to every Eulerian cycle C corresponds some oriented spanning tree rooted at v; now we will count the number of cycles corresponding to a particular tree T. This tree corresponds to those Eulerian cycles which start going from v along the fixed edge and avoid going via any edge of T as long as possible. Now we will show how to construct all such cycles.

For each vertex $u \neq v$, there are $(outdeg(u) - 1)!$ possible orderings of its outgoing edges not belonging to T (such an ordering will tell in which order the cycle should leave this vertex; the remaining edge should be the last one). For vertex v, there are again $(outdeg(v) - 1)!$ possible orderings of outgoing edges apart from the fixed one. We claim that every particular choice of all these orderings indeed determines an Eulerian cycle.

Let us start walking from v via the fixed edge; then we move according to the orderings until we get stuck. This could only happen at v; denote the obtained cycle by C. Assume that C does not pass through some edge from a vertex u; then it also does not pass along the edge from u in T. Considering a path from u to v in T, we find the edges e_1, e_2 lying in T such that $t(e_1) = s(e_2)$, and the cycle contains e_2 but not e_1. This yields that C contains all the edges outgoing from $s(e_2)$ but not all edges ingoing to this vertex; this contradicts the balancedness of G.

Thus, each tree corresponds to $\prod\limits_{u \in V(G)} (outdeg(u) - 1)!$ different Eulerian cycles, and the result follows.

Remark. As a corollary, we obtain that the number of oriented spanning trees rooted at some vertex v of a balanced strongly connected graph does not depend on the choice of v.

F.2 **Answer.** $B(k, n) = k!^{k^{n-1}}/k^n$.

The previous problem implies that $B(n, k)$ is $(k-1)!^{k^{n-1}}$ times the number of oriented spanning trees with a fixed root in the de Bruijn graph $G_{k,n}$, i.e. $B(n, k)$ is $\frac{(k-1)!^{k^{n-1}}}{k^{n-1}}$ times the total number $\tau(G(k, n))$ of oriented spanning trees in $G(k, n)$. We prove by induction on n that $\tau(G_{k,n}) = k^{k^{n-1}-1}$; this immediately provides the answer.

The base $n = 1$ is clear. Notice that $G_{k,n+1} = \mathcal{L}G_{k,n}$. Therefore, plugging ones instead of all variables onto Levine's theorem we get

$$\tau(G_{k,n+1}) = \tau(G_{k,n}) k^{(k-1)k^{n-1}},$$

which completes the inductive step.

F.3 We present a proof from the article H. Bidkhori, S. Kishore. A bijective proof of a theorem of Knuth. Combinatorics, Probability and Computing, vol. 20, is. 01, 2011; the pictures are borrowed there as well.

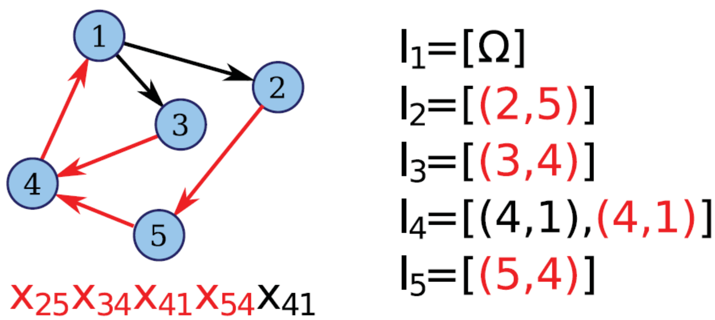
Let us expand all the brackets in both sides of the desired equality; we are to prove that the multisets of the monomials in both sides are identical. We introduce the following notion.

Definition F.3 *A tree array in a graph G is a set of ordered lists ℓ_v, one per vertex v of G, satisfying the following properties.*

a) The length of each list ℓ_v is indeg(v). All the elements of this list are edges outgoing from v (possibly, with repetitions), with the only exception: the last element of exactly one list ℓ_{v_0} is some predefined symbol Ω.

b) The last elements of all the lists ℓ_v with $v \neq v_0$ form an oriented spanning tree rooted at v_0.

The picture below depicts an example of a tree array.



We put into correspondence to every tree array a monomial containing every variable x_e with an exponent equal to the number of occurrences of e in this array. One can easily see that the tree arrays are in one-two-one correspondence with the monomials in the RHS of the required equality (moreover, the arrays where the last edges of the lists form a particular tree correspond to the monomials obtained from a monomial of this tree in $\kappa^{edge}(G)$).

So, in order to prove our equality it suffices to construct a bijection of tree arrays in G and the spanning trees in $\mathcal{L}G$ such that the monomial in $\kappa^{vertex}(\mathcal{L}G)$ of a certain tree corresponds also to its tree array. Let $\mathcal{A}$ be the set of tree arrays in G, and let $\mathcal{T}$ be the set of oriented spanning trees in $\mathcal{L}G$. We will construct algorithmically mutually inverse mappings

$$\mathbf{\Sigma} : \mathcal{A} \to \mathcal{T} \qquad \text{and} \qquad \mathbf{\Pi} : \mathcal{T} \to \mathcal{A},$$

satisfying the abovementioned condition.

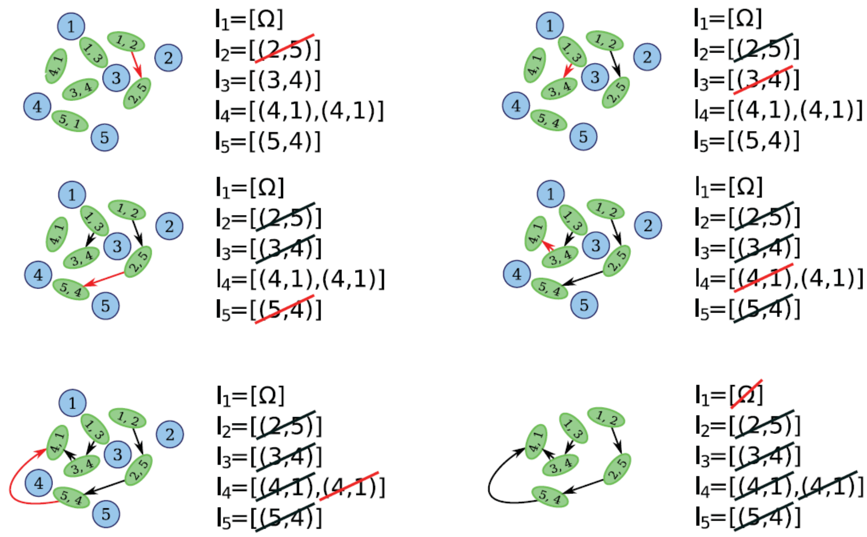
Let us fix an arbitrary enumeration of the edges of G.

Construction of $\mathbf{\Sigma}$. Take an array $A \in \mathcal{A}$. We will construct its subgraph T edge by edge, accordingly reducing array A. We start with a subgraph T with no edges; we will define $\mathbf{\Sigma}(A)$ as the obtained subgraph T when A is finally depleted. Denote the state of A and T after having performed the kth step by $A(k)$ and $T(k)$ respectively.

On the $(k+1)$th move, let us choose an edge e with the smallest number such that e is not contained in $A(k)$ and its out-degree in $T(k)$ is zero. Let g be the first item in the list $\ell_{t(e)}$ of array $A(k)$. We define $A(k+1)$ by removing g from the beginning of this list. If $g = \Omega$ then we stop, otherwise we add the edge $e \to g$ to $T(k)$ forming the subgraph $T(k+1)$. The picture below depicts the stages of this algorithm working on the previous example array.

Let G contain d edges. After k steps we have k edges having nonzero out-degrees in $T(k)$ and at most $d-1-k$ distinct edges in $A(k)$. Thus we are able to choose some edge on the $(k+1)$th step. Moreover, the list $\ell_{t(e)}$ will be nonempty, since this list decreases exactly when the in-degree of $t(e)$ decreases, so at most $indeg(t(e)) - 1$ times before this step.

We claim that T contains no oriented cycles. Indeed, if the edges $e_1 \to e_2, \ldots, e_{n-1} \to e_n)$ are already added to T, then it is impossible to add the edge $e_n \to e_1)$ completing the cycle, since e_1 was absent in A already at the moment of introducing the edge $e_1 \to e_2$.



Now let M be the tree in G consisting of the last edges on the lists of A, and let r be its root. Consider any edge $e=(u,w)$ in M (thus this edge is in ℓ_u). Assume that at some step of the algorithm the list ℓ_v becomes empty. At this moment, T has already got all its edges ending at v; in particular, the out-degree of e in T is nonzero at this moment. This means that e was chosen on some previous step. At that step, A already did not contain e, so at that step ℓ_u was empty. Thus, if ℓ_v is exhausted at some moment then ℓ_u had been exhausted before that.

Now consider the moment when ℓ_r empties; due to the arguments above, all the other lists should be empty at this moment as well. This the process will stop in exactly d steps, and the resulting subgraph $T=\mathbf{\Sigma}(A)$ is an oriented spanning tree in $\mathcal{L}G$. It is easy to see that the monomials corresponding to A and to T coincide.

Construction of $\mathbf{\Pi}$. Now take an oriented spanning tree $S\in\mathcal{T}$ rooted at r. We start with an empty array B, and we will increase this array while removing the edges from T. Denote the states of B and S after kth step by $B(k)$ and $S(k)$, respectively.

On the kth step, we consider all the vertices of $S(k)$ with the out-degree 1 and the in-degree 0, and we choose such a vertex e having the least number (recall that the vertices of $S(k)$ are edges of G). Let $e\to f$ be the edge of $S(k)$ from e. We remove this edge from S, and we add f to the end of list $\ell_{t(e)}$. When S becomes empty (which happens after $d-1$ steps), we add Ω to the end of $\ell_{t(r)}$.

When the algorithm halts, the length of every list ℓ_v in B equals $indeg(v)$. Now let M be a subgraph in G formed by all last edges in the lists ℓ_v. Assume that M contains an edge $f=(u,w)$ (lying in ℓ_u). We prove that the list ℓ_v has been completed after the list ℓ_u had been completed. Indeed, when list ℓ_w is complete, vertex f of graph S should become isolated; therefore, all the edges which ended at this vertex in the original graph S were worked out before; so f appeared in list ℓ_u or the last time before this moment.

This property yields that M contains no oriented cycles. Since each vertex of M except for $t(r)$ has exactly one out-edge, M is an oriented spanning tree. Therefore, B is a tree array, and we set $B=\mathbf{\Pi}(S)$.

It remains to show that the mappings $\mathbf{\Sigma}$ and $\mathbf{\Pi}$ are mutually inverse. Let $T=\mathbf{\Sigma}(A)$, and let the first k steps of the algorithm constructing $\mathbf{\Sigma}(\mathbf{A})$ delete from A the edges $f_1,\dots,f_k$ and insert to T the edges $e_1\to f_1)$, …, $e_k\to f_k$ in this order. We show by the induction on k that the first k steps of the algorithm constructing $B=\mathbf{\Pi}(\mathbf{T})$ insert to B the edges $f_1,\dots,f_k$ and delete from T the edges $e_1\to f_1)$, …, $e_k\to f_k)$ in the same order.

The base $k=0$ is trivial. For the step, consider the $(k+1)$th step of the $\mathbf{\Pi}$-algorithm. By the induction hypothesis, the edge $e_{k+1}\to f_{k+1}$ exists in subgraph $T(k)$. Moreover, $T(k)$ cannot contain an edge of the form $e\to e_{k+1}$, otherwise this edge would have been added to T on some step of $\mathbf{\Sigma}$ with number $\ell>k+1$; but before this ℓth step array $A(\ell-1)$ did not contain edge e_{k+1} — a contradiction. Thus e_{k+1} has required in- and out-degrees in $T(k)$.

On the other hand, if $T(k)$ contains some edge $e \to e'$ such that the in-degree of e in $T(k)$ is zero, then the number of e is greater than that of e_{k+1}; otherwise we would add the edge $e \to e'$ to T on the $(k+1)$th step of $\mathbf{\Sigma}$ instead of $e_{k+1} \to f_{k+1}$. Therefore, on the $(k+1)$th step of $\mathbf{\Pi}$ we indeed should delete edge $e_{k+1} \to f_{k+1}$ from T and add edge f_{k+1} to B.

So, we have proved that $\mathbf{\Pi}(\mathbf{\Sigma}(A)) = A$ for all $A \in \mathcal{A}$. The proof of the relation $\mathbf{\Sigma}(\mathbf{\Pi}(T)) = T$ for all $T \in \mathcal{T}$ is analogous.

Remark. The original Levine's proof of his theorem was algebraic, and it involved some technique using determinants.

G de Bruijn sequences via recurrencies

G.1 The proof of this statement involves some involved algebraic technique. If you are not acquainted with this technique, you may either postpone reading this solution for several years, or read some algebraic literature before. We use several classical facts without proof. It is known that for every positive integer n there exists a degree n polynomial f which is irreducible over the 2-element field F_2. Next, the residue classes modulo this polynomial f form a field K with 2^n elements. For every nonzero $u \in K$ the equality $u^{2^n-1} = 1$ holds; moreover, it is known that there exists a nonzero $\xi \in K$ such that $\xi^t \neq 1$ for all $0 < t < 2^n - 1$. Thus all nonzero elements of K are just the powers of ξ.

Now consider a polynomial g irreducible over F_2 such that $g(\xi) = 0$. Then g divides the polynomial $x^{2^n} - x$ which has no multiple roots (since its roots are just all elements of K). Hence, g has n distinct roots in K. We have $\deg g = n$ since $|F_2(u)| = 2^n$. So we may write $g = x^{k_1-1} + \cdots + x^{k_s-1} + x^n$, where $k_1 < k_2 < \cdots < k_s$. Now we define a template putting X exactly at the positions with numbers $k_1, k_2, \ldots, k_s$.

Next, let $\xi = \xi_1, \xi_2, \ldots \xi_n$ be the roots of g. The following system of linear equations

$$\begin{cases} x_1 + x_2 + \cdots + x_n = 0 \\ \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n = 0 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \quad \vdots \\ \xi_1^{n-2} x_1 + \xi_2^{n-2} x_2 + \cdots + \xi_n^{n-2} x_n = 0 \\ \xi_1^{n-1} x_1 + \xi_2^{n-1} x_2 + \cdots + \xi_n^{n-1} x_n = 1 \end{cases}$$

has a unique solution, since its determinant is nonzero (it is simply the Vandermonde determinant). It is easy to check now that the sequence determined by our template can be also defined as

$$a_k = x_1 \xi_1^k + \cdots + x_n \xi_n^k.$$

For every $n > 1$ we have $k_1 = 1$ since g is irreducible. One may see then that the sequence (a_k) is purely periodic, since each piece of this sequence of length n determines uniquely both the previous and the next its pieces. Let the period have the length t. The equalities $a_t = a_0$, $a_{t+1} = a_1, \ldots, a_{t+n-1} = a_{n-1}$ together with the uniqueness of the solution of the system yield $x_1 \xi_1^t = x_1, \ldots, x_n \xi_n^t = x_n$. Since one of the x_i's is nonzero, we get $\xi_i^t = 1$. Therefore, $t \geq 2^n - 1$, since the extensions $F_2(\xi_i)$ and $F_2(u)$ are isomorphic. On the other hand, we have $t \leq 2^n - 1$ since the piece of n zeroes cannot occur in our sequence, and the first repetition of the piece of n consecutive terms yields the periodicity.