Combinatorial geometry and graph colorings: from algebra to probability

A.M. Raigorodskii, with the help of V. Bulankina and A. Gusev

1 Definitions and notation

One of the most famous and fascinating objects of combinatorial geometry is *the chromatic number of a space*. Before we introduce it, we remind that the space $\mathbb{R}^n$, which is called *the n-dimensional Euclidean space*, is just the set of all "points" $\mathbf{x}$, each of which is a sequence consisting of n real numbers: $\mathbf{x} = (x_1, \ldots, x_n)$. Moreover, between any two points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, one can find the distance using the formula

$$|\mathbf{x} - \mathbf{y}| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}.$$

In particular, for $n = 1$, we get the usual line, for $n = 2$ — the usual plane, for $n = 3$ — the usual space.

The chromatic number of $\mathbb{R}^n$ is the quantity denoted by $\chi(\mathbb{R}^n)$ and equal to the minimum number of colors needed to color all the points of the space $\mathbb{R}^n$, so that the distance between any two points of the same color is not 1.

We will start from the simplest facts, which are widely known, and we will finally come to advanced results obtained just few months before the Summer Conference. Moreover, the methods, which we shall study, will be very different and nontrivial varying from linear algebra to probability theory and random graphs.

2 Problems before the intermidiate finish

2.1 The simplest bounds for the chromatic number

Problem 1. Prove that $\chi(\mathbb{R}^1) = 2$.

Problem 2. Prove that $\chi(\mathbb{R}^2) \geqslant 4$.

Problem 3. Prove that $\chi(\mathbb{R}^2) \leqslant 7$.

Problem 4. Prove that $\chi(\mathbb{R}^3) \leqslant 27$.

Problem 5. Prove that $\chi(\mathbb{R}^3) \geqslant 5$.

Problem 6. Prove that $\chi(\mathbb{R}^n)$ is finite for every n.

Problem 7*. Prove that $\chi(\mathbb{R}^n) \leqslant (\lceil \sqrt{n} \rceil + 1)^n$.

Problem 8. Prove that in $\mathbb{R}^n$ there is a set of $n+1$ points, whose pairwise distances are equal to 1, and therefore, $\chi(\mathbb{R}^n) \geqslant n+1$.

Problem 9*. Prove that $\chi(\mathbb{R}^n) \geqslant n+2$.

2.2 Distance graphs of special type, their simplest properties, and the connection with the chromatic number of a space

Recall that the scalar product of vectors $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ is the expression

$$(\mathbf{x}, \mathbf{y}) = x_1 y_1 + \ldots + x_n y_n.$$

One can easily check that always

$$|\mathbf{x} - \mathbf{y}|^2 = (\mathbf{x}, \mathbf{x}) + (\mathbf{y}, \mathbf{y}) - 2(\mathbf{x}, \mathbf{y}). \tag{1}$$

Let r, s be some natural numbers. For each $n \in \mathbb{N}$ denote by $G(n, r, s)$ the graph, whose set of vertices is

$$V(n, r) = \{\mathbf{x} = (x_1, \ldots, x_n) : \ x_i \in \{0, 1\}, \ x_1 + \ldots + x_n = r\}$$

and whose set of edges is

$$E(n, r, s) = \{\{\mathbf{x}, \mathbf{y}\} : \ (\mathbf{x}, \mathbf{y}) = s\}.$$

In other words, vertices are all possible vectors consisting of 0s and 1s, such that in every such vector one has exactly r 1s and $n - r$ 0s. In turn, by edges those and only those vertices are joined whose scalar product equals s. Due to formula (1) one can say that edges are those and only those pairs of vertices whose distance equals $\sqrt{2r - 2s}$. This is why the graphs $G(n, r, s)$ are called *distance graphs.*

It is also convenient to have the following interpretation of a graph $G(n, r, s)$. Its vertices can be considered as all possible r-element subsets of the set $\mathcal{R}_n = \{1, 2, \ldots, n\}$. Its edges can be considered as pairs of subsets whose intersections have cardinalities equal to s. Please make sure that you understand it!

Recall that an *independent set* of vertices of a graph is a set, in which every two vertices are not joined by an edge. The *independence number* $\alpha(G)$ of a graph G is the number of vertices in any maximal (by cardinality) independent set. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors needed to color all the vertices of the graph in such a way that between any two vertices of the same color, there are no edges.

Problem 10. Prove that for any n, r, s, one has $\chi(\mathbb{R}^n) \geqslant \chi(G(n, r, s))$.

Problem 11. Prove that for every graph $G = (V, E)$, one has $\chi(G) \geqslant \frac{|V|}{\alpha(G)}$.

Problem 12. Find $\alpha(G(n, 3, 1))$. Derive from the obtained result a considerable improvement to the bound in the problem 9.

Problem 13*. Find $\chi(G(n, 3, 1))$ for $n = 2^k$. **Hint.** Use the problems 11 and 12 as well as the following lemma and induction by k.

Lemma 1. *Let n be an even number and P_n be the set of all unordered pairs $\{a, b\}$ of natural numbers both of which do not exceed n. Then there exist such sets of pairs $B_1, \ldots, B_{n-1}$ that*

$$P_n = B_1 \sqcup \ldots \sqcup B_{n-1}.$$

Moreover, for any $i = 1, \ldots, n-1$, no two pairs from B_i contain a common element. For odd n, we have a partition

$$P_n = B_1 \sqcup \ldots \sqcup B_n,$$

and, again for any $i = 1, \ldots, n$, no two pairs from B_i contain a common element.

Problem 14.** Find sharpest possible bounds (ideally — a formula) for $\chi(G(n,3,1))$ for every n.

Problem 15. Prove that $\alpha(G(n,r,s)) \geqslant C_{n-s-1}^{r-s-1}$.

Problem 16*. Prove that $\alpha(G(n,r,0)) = C_{n-1}^{r-1}$, if $2r \leqslant n$.

Problem 17. Prove that $\chi(G(n,r,0)) \leqslant n - 2r + 2$, if $2r \leqslant n$.

Problem 18. Prove that $\chi(G(n,r,s)) \leqslant C_r^s C_{n-r}^{r-s} + 1$.

Problem 19. Prove that $\chi(G(n,r,s)) \leqslant C_n^{s+1}$.

Problem 20*. Let $k = \left[\frac{r-1}{s}\right]$. Prove that $\chi(G(n,r,s)) \leqslant k \cdot C_{\lceil \frac{n}{k} \rceil}^{s+1}$.

Problem 21*. Prove that $n - r + 1 \leqslant \chi(G(n,r,r-1)) \leqslant n$ for $n = 2^k$. **Hint.** Use Lemma 1 and induction by r and k.

Problem 22.** Find $\chi(G(n,r,r-1))$ or at least refine the bounds from the problem 21.

Please find out that none of the results that you have obtained does not allow you to improve the lower bounds of the value $\chi(\mathbb{R}^n)$ found in the problems 9, 12. In view of the problem 11, it be good to study upper bounds for the independence numbers of the graphs $G(n,r,s)$. It turns out that many of such bounds can be obtained with the help of the linear algebra method. Thus, in the next section, we will recall some basic notions of linear algebra.

2.3 Basics of linear algebra and its applications

We say that vectors $\mathbf{x}_1, \ldots, \mathbf{x}_t$ in $\mathbb{R}^n$ are *linearly independent*, if the equality $c_1\mathbf{x}_1 + \ldots + c_t\mathbf{x}_t = 0$ is possible only in the case when $c_1 = \ldots = c_t = 0$.

Problem 23. Prove that the maximum number of linearly independent vectors in $\mathbb{R}^n$ equals n.

Problem 24. Prove that if $\mathbf{x}_1, \ldots, \mathbf{x}_n$ form an arbitrary system of linearly independent vectors in $\mathbb{R}^n$, then any vector $\mathbf{x} \in \mathbb{R}^n$ can be represented as $\mathbf{x} = c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n$, where $c_1, \ldots, c_n$ are real numbers. (The system $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is called a *basis* of the space and the expression $c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n$ is called *linear combination* of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ with *coefficients* $c_1, \ldots, c_n$. In these terms, any vector $\mathbf{x} \in \mathbb{R}^n$ can be represented as a linear combination of the vectors of the basis.)

Let p be a prime number. Let $\mathbb{Z}_p$ be the set of congruences modulo p. The space $\mathbb{Z}_p^n$, similarly to $\mathbb{R}^n$, is just the set of all the sequences of numbers from $\mathbb{Z}_p$. The operations of the sum of "vectors" $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_p^n$ and of their product with elements of $\mathbb{Z}_p$ is done, as usual, coordinate by coordinate, but, this time, every coordinate is taken modulo p.

The notions of linear independence and of a basis for $\mathbb{Z}_p^n$ are defined in the same way as for $\mathbb{R}^n$. However, here all the numbers c_i are elements of $\mathbb{Z}_p$ — not $\mathbb{R}$, — and the equality to zero is understood as the equality to zero modulo p.

Problem 25. Prove that the maximum number of linearly independent vectors in $\mathbb{Z}_p^n$ equals n and that any maximal system forms a basis.

Problem 26. Let $W = \{\mathbf{x}_1, \ldots, \mathbf{x}_t\}$ be an arbitrary independent set of vertices of the graph $G(n, 3, 1)$. Prove that the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_t$ are linearly independent in $\mathbb{Z}_2^n$ and thus $\alpha(G(n, 3, 1)) \leqslant n$, which is only by an "epsilon" weaker than the result of the problem 12!

Let $F \in \{\mathbb{R}, \mathbb{Z}_p\}$. Let $x_1, \ldots, x_n$ be "variables". By a *monomial* depending on these n variables we mean an expression of the form $x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$, where $a_1, \ldots, a_n$ are some non-negative integers. A *polynomial* is an arbitrary linear combination of monomials. More precisely, a polynomial P belongs to $F[x_1, \ldots, x_n]$, if its coefficients are from F. Polynomials are added and multiplied according to the usual rules. Also if $P \in F[x_1, \ldots, x_n]$, then it can be multiplied by any element of F. In any case, the rules of summation and multiplication of the coefficients of polynomials are defined by the rules of summation and multiplication in the set F. The *degree* of a monomial is the sum of the degrees of its variables. The degree of a polynomial is the maximum of the degrees of its monomials. A polynomial $P \in F[x_1, \ldots, x_n]$ *equals zero*, if all its coefficients are equal to zero in F. Polynomials $P_1 \in F[x_1, \ldots, x_n], \ldots, P_t \in F[x_1, \ldots, x_n]$ are *linearly independent* over F, if $c_1 P_1 + \ldots + c_t P_t = 0$ only in the case when all the numbers $c_1 \in F, \ldots, c_t \in F$ are equal to zero in F. It is obvious that any polynomial is generated by the *basis* consisting of its monomials.

Problem 27. Prove that if some polynomials are linearly independent over their F, then their number does not exceed the number of monomials in a basis, which generates all these polynomials.

Problem 28. Let $W = \{\mathbf{x}_1, \ldots, \mathbf{x}_t\}$ be an arbitrary independent set of vertices of the graph $G(n, 5, 2)$. Let polynomials $P_1 \in \mathbb{Z}_3[y_1, \ldots, y_n]$, $P_2 \in \mathbb{Z}_3[y_1, \ldots, y_n]$, $\ldots$, $P_t \in \mathbb{Z}_3[y_1, \ldots, y_n]$ be given by formulae

$$P_i(\mathbf{y}) = P_i(y_1, \ldots, y_n) = (\mathbf{x}_i, \mathbf{y})((\mathbf{x}_i, \mathbf{y}) - 1), \quad i = 1, \ldots, t.$$

For example, if $\mathbf{x}_1 = (1, 1, 1, 1, 1, 0, \ldots, 0)$, $\mathbf{x}_2 = (0, \ldots, 0, 1, 1, 1, 1, 1)$, then

$$P_1(y_1, \ldots, y_n) = (y_1 + y_2 + y_3 + y_4 + y_5)(y_1 + y_2 + y_3 + y_4 + y_5 - 1) = y_1^2 + \ldots + y_5^2 + 2y_1y_2 + \ldots + 2y_4y_5 - y_1 - \ldots - y_5,$$

$$P_2(y_1, \ldots, y_n) = (y_{n-4} + y_{n-3} + y_{n-2} + y_{n-1} + y_n)(y_{n-4} + y_{n-3} + y_{n-2} + y_{n-1} + y_n - 1) =$$
$$= y_{n-4}^2 + \ldots + y_n^2 + 2y_{n-4}y_{n-3} + \ldots + 2y_{n-1}y_n - y_{n-4} - \ldots - y_n.$$

Prove that the polynomials $P_1, \ldots, P_t$ are linearly independent over $\mathbb{Z}_3$ and therefore $\alpha(G(n, 5, 2)) \leqslant C_n^2 + 2C_n^1$.

Problem 29. Assume that in the conditions of the previous problem the polynomials P_i are substituted by P_i' according to the following rule: every monomial of the form y_i^2 is changed by y_i, and after that monomials of the same form are added. Prove that the polynomials $P_1', \ldots, P_t'$ corresponding to the vectors from an independent set of vertices W of the graph $G(n, 5, 2)$ are also linearly independent over $\mathbb{Z}_3$, similarly to the initial polynomials $P_1, \ldots, P_t$. Derive from this fact the bound $\alpha(G(n, 5, 2)) \leqslant C_n^2$ and compare it with the bound from the problem 15.

Problem 30. Derive from the result of the previous problem a lower bound for $\chi(\mathbb{R}^n)$, which considerably improves the bound from the problem 12. Ensure yourselves, however, that, in view of the problem 19, substantial further advances based on the graph $G(n, 5, 2)$ cannot be done.

Problem 31. Let $W = \{\mathbf{x}_1, \ldots, \mathbf{x}_t\}$ be an arbitrary independent set of vertices of the graph $G(n, 9, 4)$. Let polynomials $P_1 \in \mathbb{Z}_5[y_1, \ldots, y_n]$, $P_2 \in \mathbb{Z}_5[y_1, \ldots, y_n]$, $\ldots$, $P_t \in \mathbb{Z}_5[y_1, \ldots, y_n]$ be given by the formulae

$$P_i(\mathbf{y}) = P_i(y_1, \ldots, y_n) = (\mathbf{x}_i, \mathbf{y})((\mathbf{x}_i, \mathbf{y}) - 1)((\mathbf{x}_i, \mathbf{y}) - 2)((\mathbf{x}_i, \mathbf{y}) - 3), \quad i = 1, \ldots, t.$$

Prove that the polynomials $P_1, \ldots, P_t$ are linearly independent over $\mathbb{Z}_5$.

Problem 32. Which upper bound for $\alpha(G(n, 9, 4))$ follows from the previous problem?

Problem 33. Let in the conditions of the problem 31 the polynomials P_i be replaced by some P_i' according to the following rule: in them, every monomial, which appears after opening the brackets and summing similar monomials, has of course the form $y_1^{a_1} \cdot \ldots \cdot y_n^{a_n}$; if among the numbers a_i, one has some numbers greater than or equal to 2, then we replace all of them by 1s and sum up similar monomials. For example, the monomial $y_1^2 y_2^2$ is transformed to $y_1 y_2$ and the same is true for the monomials $y_1^2 y_2, y_1 y_2$, etc. Prove that the polynomials $P_1', \ldots, P_t'$ corresponding to the vectors from an independent set W of the graph $G(n, 9, 4)$ are also linearly independent over $\mathbb{Z}_5$, just as it was with the initial polynomials $P_1, \ldots, P_t$. Derive from this fact the bound $\alpha(G(n, 9, 4)) \leqslant C_n^4 + C_n^3 + C_n^2 + C_n^1 + C_n^0$ and compare it with the bound from the problem 15.

Problem 34. Derive from the result of the previous problem a lower bound for $\chi(\mathbb{R}^n)$, which considerably refines the bound from the problem 30. Ensure yourselves, however, that, in view of the problem 19, substantial further advances based on the graph $G(n, 9, 4)$ cannot be done.

Problem 35. Let r and s be such that $r - s = p$, where p is a prime number and $r - 2p < 0$. Prove that $\alpha(G(n, r, s)) \leqslant \sum\limits_{k=0}^{p-1} C_n^k$. Compare this bound with the bound from the problem 15.

Problem 36*. Explore lower bounds for the quantity $\chi(\mathbb{R}^n)$, which follow from the results of the previous problem. How do these bounds correlate with the bounds from the problem 19?

3 Problems after the intermediate finish

Before the intermediate finish we have seen how important are the independence numbers of graphs for obtaining lower bounds on the chromatic number of a space. Moreover, we considered different sequences of distance graphs — the sequences $\{G(n, r, s)\}_{n=1}^{\infty}$ with given r and s. It is interesting to understand how will be changed the independence numbers, if, instead of the graphs $G(n, r, s)$, we take their spanning subgraphs, i.e., if we take the same sets of vertices and partially remove the edges. It seems to be evident that the independence numbers must become substantially larger, provided we remove many edges. However, surprisingly, sometimes this is true, but sometimes this is completely wrong! In order to obtain the corresponding results, let us learn some random graphs and some probability theory.

3.1 The random Erdős–Rényi graph and some of its probabilistic characteristics

Let $V_n = \{1, \ldots, n\}$ be a set of vertices. In principle, one can construct on this set C_n^2 edges, if we do not allow multiple edges, loops and orientation. Let us each of these C_n^2 potential edges draw with *probability* $p \in [0, 1]$, which is the same for every edge. Different edges appear independently. Let $G = (V_n, E)$ be a graph which can appear as a result of the just-described probabilistic procedure. Denote the probability of its appearance by $\mathbb{P}(G)$. Clearly it is equal to $p^{|E|}(1-p)^{C_n^2-|E|}$. If A is a property of a graph, then its probability — $\mathbb{P}(A)$ — is the sum over all the graphs G, which have property A, of the probabilities of these graphs.

Denote by Ω_n the set of all graphs on the vertices V_n. Any function X defined on Ω_n and taking real values is called *random variable*. For example, the number of triangles in a graph or its number of connected components, or its independence number are random variables. Here you must understand that the variables are random only due to the fact that apriori we do not know which graph will (randomly) come to us. When a graph is already born the value of X is determined uniquely!

Any random variable has some "mean value" — the so-called *(mathematical) expectation*. The expectation of a variable X is the number $\mathbb{M}X$ defined by the formula $\mathbb{M}X = \sum\limits_{G \in \Omega_n} X(G)\mathbb{P}(G)$. We just sum up the values of the function X on all the graphs multiplied by the probabilities of these graphs. Of course this is some weighted average (weights are probabilities) — a kind of the mass centre). Let us learn to calculate expectations and apply the obtained results to the study of the properties of random graphs.

Problem 37. Prove that if a random variable is just a constant c, then $\mathbb{M}c = c$.

Problem 38. Let X_1, X_2 be random variables and c_1, c_2 be some fixed numbers. Certainly $c_1X_1 + c_2X_2$ is also a random variable. Prove that its expectation equals $c_1\mathbb{M}X_1 + c_2\mathbb{M}X_2$. This property is called *linearity of expectation.*

Problem 39. Using linearity of expectation find expectations of a) the number of triangles in the random graph; b) the number of connected components of the random graph, each of which is a cycle on k vertices (k is a fixed given number); c) the number of independent sets in the random graph, each of which has cardinality k (k is a fixed given number).

Problem 40. Prove *Markov's inequality*: if X is a random variable taking non-negative values and a is a positive number, then $\mathbb{P}(X \geqslant a) \leqslant \frac{\mathbb{M}X}{a}$.

Problem 41. Using Markov's inequality prove that if $p = \frac{1}{2}$, then $\mathbb{P}(\alpha(G) \leqslant 2\log_2 n) \to 1$ as $n \to \infty$ (it is used to say that "almost surely $\alpha(G) \leqslant 2\log_2 n$").

Note that actually for $p = \frac{1}{2}$ almost surely $\alpha(G) \geqslant (2-\varepsilon)\log_2 n$ for any given arbirarily small $\varepsilon > 0$. More precisely, — and we shall use it later, — the following theorem holds.

Theorem 1. *For any $\varepsilon > 0$ and large enough n the inequality*

$$\mathbb{P}(\alpha(G) \geqslant (2-\varepsilon)\log_2 n) > 1 - 2^{-n}$$

holds.

Thus, for $p = \frac{1}{2}$, almost surely the independence number is approximately $2\log_2 n$. In other words, what's happening? We take the complete graph on n vertices and we delete some of its edges, each with probability $\frac{1}{2}$. A typical graph, which appears as a result of this procedure, has about $\frac{C_n^2}{2}$ edges — two times less than the complete graph has. And the independence number of a typical graph is in a logarithm times greater than the independence number of the initial graph (the independence number of the complete graph is of course 1). OK, the number of edges is two times smaller and the independence number increases: quite natural. It turns out that for some $G(n,r,s)$ when deleting their edges randomly, we get the same "expected" result (the independence number becomes approximately log times bigger). But the miracle is that this happens not everytime! For many $G(n,r,s)$ the independence number does not change at all! Below, we will study together examples of both situations.

3.2 Random subgraphs of the graph $G(n,3,0)$

Let $G_{1/2}(n,3,0)$ be a random subgraph of the graph $G(n,3,0)$ obtained by deleting mutually independently some edges of the graph $G(n,3,0)$, each with probability $1/2$.

Problem 42. If you solved the problem 16, then just derive from it that $\alpha(G(n,3,0)) = C_{n-1}^2$. Otherwise, try to solve this particular case of that problem.

In a series of problems below, we will prove that almost surely $\alpha(G_{1/2}(n,3,0)) \leqslant C_{n-1}^2\left(1+\frac{1}{\ln n}\right)$. This is exactly what we mentioned above speaking about a surprising phenomenon: there is no growth in log times; if there is some growth, then it is only in such a number of times that itself tends to 1 as $n \to \infty$! One can prove even stronger facts, but this is very difficult, and we do not want to do it here: we just want to see the essence!

In what follows, we will for conciseness omit integer parts around quantities which must be integer. For example, writing $C_{\log_2 n}^k$ means, depending on context, that actually we take an upper or a lower integer part of the number $\log_2 n$. None of the computations will become false after this roughening.

Problem 43. Put $k = C_{n-1}^2\left(1+\frac{1}{\ln n}\right)$. Assume that k is integer (cf. a remark before the problem). Let $A \subset V(n,3)$ be an arbitrary set of vertices of the graph $G(n,3,0)$ having cardinality k. Denote by $r(A)$ the number of edges of the graph $G(n,3,0)$, whose both ends are in A. Since $|A| = k > \alpha(G(n,3,0))$, it is clear that $r(A) > 0$. Let X_k be the random variable equal to the number of independent sets of size k in a graph $G_{1/2}(n,3,0)$. Prove that

$$\mathbb{M}X_k = \sum_{A\subset V(n,3):\ |A|=k} \left(\frac{1}{2}\right)^{r(A)}. \tag{2}$$

Problem 44. Prove that our aim will be attained, if we will prove that $\mathbb{M}X_k \to 0$ as $n \to \infty$.

Clearly we have to learn bounding *from below* the quantity $r(A)$. For each A, denote by $B = B(A)$ any (chosen for everytime) subset of A, which is independent in $G(n,3,0)$ and has the maximum cardinality among all analogous subsets of the set A.

Problem 45. Let $A \subset V(n,3)$, $|A| = k$. Let $B = B(A)$. Note that $k \approx \frac{n^2}{2}$. Assume that $|B|$ is substantially smaller than k: for example, let $|B| < n^{1.9}$ (here the strange number 1.9 is taken almost arbitrarily; important is that it is strictly smaller than 2). Prove that for large enough n the inequality $r(A) \geqslant \frac{k^2}{3|B|}$ holds (being more careful one can replace 3 in the denominator by "almost" 2, but does not matter).

Problem 46. Let $A \subset V(n,3)$, $|A| = k$. Let $B = B(A)$. Let $|B| > 9n$ or, even better (in addition to the previous problem), $|B| \geqslant n^{1.9}$. Prove that $r(A) \geqslant (|B| - 9n)(|A| - |B|)$.

Problem 47. Decompose sum (2) into two parts: in the first part, only those A will be taken, for which $|B| < n^{1.9}$; in the second part, all the other sets A will be considered. To the summands in both parts apply the bounds from the corresponding problems and make it sure that the whole sum (2) does really tend to zero, which means that we are done!

How can we change the threshold $n^{1.9}$ in order to still get the same result?

3.3 Random subgraphs of the graph $G(n,3,1)$

Let $G_{1/2}(n,3,1)$ be a random subgraph of the graph $G(n,3,1)$ obtained by deleting mutually independently some edges of the graph $G(n,3,1)$, each with probability $1/2$. It seems that everything is the same as with the graphs $G(n,3,0)$. This is completely wrong!

Recall that $\alpha(G(n,3,1)) \approx n$ (see the problem 12).

Problem 48*. Write down an analog of inequality (2) and prove an analog of the bound from the problem 45. Ensure yourselves eventually that there exists a $c > 0$, with which almost asurely $\alpha(G_{1/2}(n,3,1)) \leqslant cn\log_2 n$.

Problem 49. Prove that in the graph $G(n,3,1)$, there exist approximately $\frac{n}{2}$ complete subgraphs, each having about $\frac{n}{4}$ vertices, such that every two of them are not connected by an edge.

Problem 50. With the help of the previous problem and Theorem 1 prove that for every $\varepsilon > 0$, almost surely $\alpha(G_{1/2}(n,3,1)) \geqslant (1-\varepsilon)n\log_2 n$.

Thus, for $G_{1/2}(n,3,1)$, just as it was for the Erdős–Rényi random graph, we again have the growth of the independence number in approximately logarithm of the number of vertices times!

Problem 51*. Try to improve into a constant number of times the bound in the problem 50.

3.4 Random subgraphs of the graph $G(n,2,1)$

Problem 52. Find $\alpha(G(n,2,1))$.

Problem 53. Let $r(A)$ be the same as in formula (2). Prove that always $r(A) \geqslant \frac{2|A|^2}{n} - |A|$.

Problem 54. Derive from the previous problem the inequality $\alpha(G_{1/2}(n,2,1)) \leqslant \left(\frac{1}{2}+\varepsilon\right) n\log_2 n$, which is true for every $\varepsilon > 0$ almost surely.

Problem 55*. Prove a lower bound for $\alpha(G_{1/2}(n,2,1))$ which has order of magnitude $cn\log_2 n$ with some $c > 0$.

Problem 56.** Find a constant c in the assertion: for any $\varepsilon > 0$, almost surely

$$(c-\varepsilon)n\log_2 n \leqslant \alpha(G_{1/2}(n,2,1)) \leqslant (c+\varepsilon)n\log_2 n.$$

References

[1] A.M. Raigorodskii, *The chromatic numbers*, Moscow Centre for Continuous Mathematical Education (MCCME), Moscow, Russia, 2003 (in Russian).

[2] A.M. Raigorodskii, *The linear algebra method in combinatorics*, Moscow Centre for Continuous Mathematical Education (MCCME), Moscow, Russia, 2007 (in Russian).

[3] A.M. Raigorodskii, *Probability and algebra in combinatorics*, Moscow Centre for Continuous Mathematical Education (MCCME), Moscow, Russia, 2008 (in Russian).

[4] A.M. Raigorodskii, *Models of random graphs*, Moscow Centre for Continuous Mathematical Education (MCCME), Moscow, Russia, 2011 (in Russian).