A square from similar rectangles

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We call a number r *tilible* if there exists a square that can be tiled by rectangles whose ratio of length of the two sides is r. Obviously, if r is tilible then $1/r$ is also tilible.
We first go through Problem 2,3,4,5,6,8,9.

Problem 2.

A. Assume that the ratio of rectangles in Figure 1 is b, then the ratio of the window is $\frac{b}{2}+\frac{1}{b}$. So if the window is a square, we can get that $b^2-2b+2=0$, which has no real root. Hence the answer is yes.

B. Assume that the ratio of rectangles in Figure 1 is b, then the ratio of the window is $\frac{b}{3}+\frac{1}{2b}$. So if the window is a square, we can get that $2b^2-6b+3=0$, which has positive root $\frac{3\pm\sqrt{3}}{2}$. Hence the answer is no.

Problem 3.

Consider the polynomial $f(x)=x^3-x+2x^2-1$, since $f(0)=-1<0$ and $\lim_{x\to+\infty} f(x)=+\infty>0$, there exists a positive root r of $f(x)$.

Now, we claim that one can tile a square by rectangles whose ratio of two lengths is r. In Figure 3, it's easy to see that each rectangle has ratio r, and since $r^3-r^2+2r-1=0$, we know that $1+r^2=r^3+2r$, so Figure 3 is a square. Notice that rectangles in Figure 3 aren't congruent, so we're done. Hence the answer is yes.

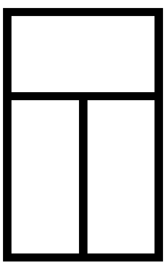

Figure 1

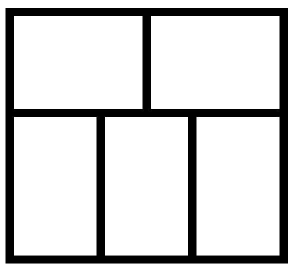

Figure 2

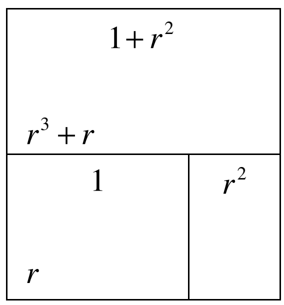

Figure 3

Problem 4.

The answer is no. Otherwise, assume that square *ABCD* is tiled by 5 squares and its side length is 1. If a square covered at least two points of *A*, *B*, *C*, *D*, it should cover all of them, hence *ABCD* is tiled by only 1 square, a contradiction.

So there are four distinct squares cover the point *A*, *B*, *C*, *D*, respectively. Let square *a* be the square that cover the point A, etc.

If square *a* and square *b* don't touch each other, then the remain square *e* must cover the area between square *a* and *b*. Now, if square *d* doesn't touch square *a*, then there is no square remain to cover the area between square *a* and *d*. Therefore, square *d* touches square *a*, and square *c* touches square *b*. Assume that s_a is the side length of square *a*, etc. Since square *a* doesn't touch square *b*, and square *d* touches square *a* , square *c* touches square *b* , we can get that $s_a + s_b < 1, s_a + s_d = 1, s_b + s_c = 1$, so $s_c + s_d > 1$. Hence square *c* and *d* have a common region, a contradiction. Hence square *a*, *b*, *c*, *d* touch each other, and then $s_a + s_b = s_b + s_c = s_c + s_d = 1$, which gives us $s_a = s_c, s_b = s_d$. Hence $1 = s_a^2 + s_b^2 + s_c^2 + s_d^2 + s_e^2$

$> s_a^2 + s_b^2 + s_c^2 + s_d^2 = 2s_a^2 + 2s_b^2 \geq (s_a + s_b)^2 = 1$, also a contradiction. In conclusion, it's impossible to tile a square by 5squares.

Problem 5.

C. Assume that a is the side length of square A in Figure 4, etc. W.L.O.G. $b = 1$, then it's easy to get that $c = d+1, a = d+2, e = d+3, f = 4, g = 2d+1, h = 3n-3$, since $e + f = i = h - f$, we know that $d = 7$, so the big rectangle we get has side length 32 and 33, which is not a square. So the board isn't square.

D. Let the side lengths of 21 small square in figure D be 2,4,7, 8,9, 11,15,16,17,18,19,24,25,27,29,33,35,37,42,50, one can show that the big rectangle is also a square.

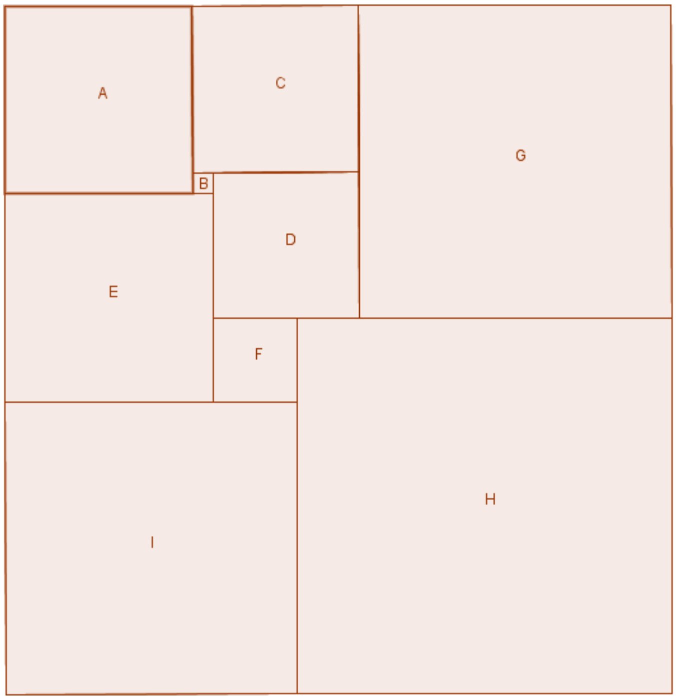

Figure 4

Problem 6.

The answer is yes. From Figure D, we know that 21 squares whose side length is 1, 2,4,7, 8,9, 11,15,16,17,18,19,24,25,27,29,33,35,37,42,50 can tile a square. Assume that they tile a square of side length x, then it's easy to know that by adding 21 squares whose side length is $2x$, $4x$, 7x, …, $50x$, we can get a square of side length x^2. Continue this operation, and we're done.

Problem 8.

The answer is no.

If rational numbers a_1, a_2, b_1, b_2 satisfy $a_1 + b_1\sqrt{2} = a_2 + b_2\sqrt{2}$, then

$(b_1 - b_2)\sqrt{2} = a_1 - a_2$. Since $\sqrt{2}$ isn't rational, we have $a_1 = a_2$ and $b_1 = b_2$. So

$1+\sqrt{2}=(a+b\sqrt{2})^2=(a^2+2b^2)+2ab\sqrt{2}$ implies

$a^2+2b^2=1$, $ab=1/2$. So $a^2-2ab+2b^2=0,(a-b)^2+b^2=0,a=b=0$, a contradiction.

Problem 9.(De Bruijn's Theorem)

We claim that a rectangle can be tiled by rectangles with one of the sides equal to 1 if and only if it has at least one side whose length is a positive integer.

First, we prove that if a rectangle can be tiled by rectangles with one of the sides equal to 1, than it has at least one side whose length is a positive integer. Dissect the plane into squares of side length 1/2 and color them like a checkerboard. Set one of the big rectangle's vertices at (0,0), and orient its sides to be parallel to the checkerboard's grid lines. It's easy to see that every small rectangle covers equal area of black and white, so the big rectangle must have equal area of black and white, too.

Now, assume that the big rectangle's two sides have length a and b, our target is to prove that $\{a\}=0$ or $\{b\}=0$. If it doesn't happen, W.L.O.G. $\{a\}>\{b\}$. It's easy to see that when we move away the rectangle ($[a]$,$[b]$), ($[a]$,b), (a,$[b]$), (a,b), the remain region has equal area of black and white. So the rectangle ($[a]$,$[b]$), ($[a]$,b), (a,$[b]$), (a,b) has equal area of black and white, hence a $\{a\}\times\{b\}$ rectangle cover same area of black and white. W.L.O.G. ($[a]$,$[b]$) is in the black squares.

Case 1 $0 < \{b\} < \{a\} < 1/2$

In this case, the whole $\{a\}\times\{b\}$ rectangle is black, a contradiction.

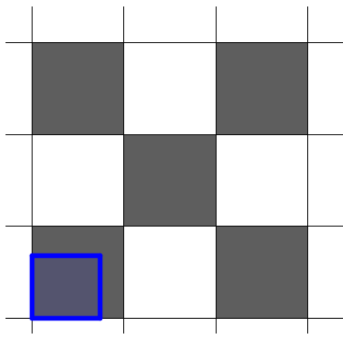

Figure 5

Case 2 $0 < \{b\} < 1/2 < \{a\} < 1$

We can “fold” the rectangle like Figure 6 shown, and this case is reduced to Case 1, a contradiction.

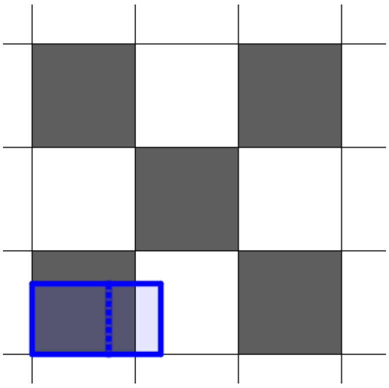

Figure 6

Case 3 $1/2 < \{b\} < \{a\} < 1$

We can fold the rectangle like Figure 7 shown, and this case is reduced to Case 2, also a contradiction.

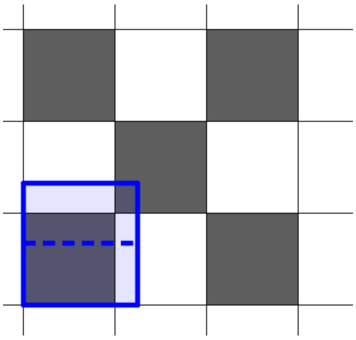

Figure 7

In conclusion, there must be at least one 0 between $\{a\}$ and $\{b\}$, so the big rectangle has at least one side whose length is positive integer.

Last, we prove that if the rectangle has at least one side whose length is a positive integer, then it can be tiled by rectangles with one of the sides equal to 1. Assume that it has a side whose length is a positive integer n, then dividing the rectangle into n equal rectangles satisfies the condition, and we’re done.

Back to the main problem: When is it possible to tile a square by rectangles similar to a given one?

We can get a very easy result first.

Theorem 1 Given a positive rational number p, then it is possible to form a rectangle of ratio r by some congruent rectangles of ratio pr.

Proof Assume that $p=a/b$, where a and b are positive integers, then we can put a rows and b columns of congruent rectangles of ratio pr, form a rectangle of ratio r, which is what we want.

Plugging r by 1 and p by m/n, we can solve Problem 1.

From Theorem 1, we know that if a square is tiled by some rectangles of ratio r, then we can tile each small rectangle to some rectangles of ratio pr. So we tile a square by some rectangles of ratio pr successfully. Notice that $1/p$ is also a positive rational number, we know that:

Corollary 1 r is tilible if and only if pr is tilible for all positive rational number p.

Since 1 is tilible, so from Corollary 1, we can get that

Theorem 2 All positive rational numbers are tilible.

Next, we prove a strong statement. For a set S, where S is a set of positive number, we define four operations:

Operation 1 If $a, b \in S$, then put $a+b$ into S.

Operation 2 If $a, b \in S$, then put $\frac{a+b}{ab}$ into S.

Operation 3 If $a, b, c \in S$, then put $\frac{ab}{a+b+c}, \frac{bc}{a+b+c}, \frac{ca}{a+b+c}$ into S.

Operation 4 If $a, b, c \in S$, then put $a+b+\frac{ab}{c}, b+c+\frac{bc}{a}, c+a+\frac{ca}{b}$ into S.

Then we have the following result.

Theorem 3 If a rectangle of ratio k can be tiled by some rectangles of ratio r, then we can do finitely many operations to $S=\{r, 1/r\}$ to make k be in S.

Proof Define R_a by $R_a = \frac{h_a}{v_a}$, where p_a is a rectangle, h_a is the length of the horizontal side of p_a, and v_a is the length of the vertical side of p_a. Let P be the set of all small rectangles. It's easy to show that $R_a \in \{r, \frac{1}{r}\} \forall p_a \in P$.

Now, consider a graph $G(v,e)$, where there is one vertex corresponding to each vertical line that is a side of some rectangles, and there is one edge e corresponding to each rectangle p_e. An edge connects the vertices corresponding to its rectangle's left and right side. Figure 8 is an example.

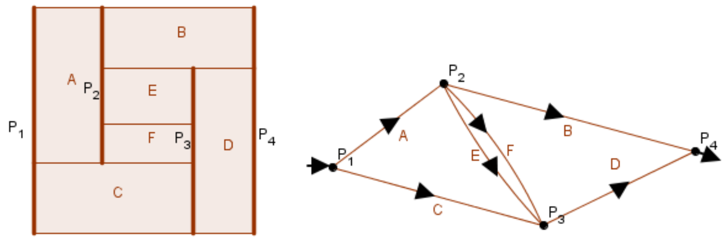

Figure 8

Now, if there is a vertex v whose degree is 2, and the edges incident to vertex v are a and b, which correspond to rectangles p_a, p_b. We merge the rectangles p_a and p_b to a rectangle p_s. It's easy to show that $R_s = R_a + R_b$. We call this operation *Combination 1*.

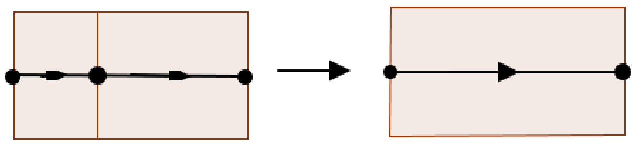

Figure 9

If there are two multiple edges a and b between two vertices v_1, v_2, we merge the rectangles p_a and p_b to a rectangle p_s. It's easy to show that $R_s = \frac{R_a R_b}{R_a + R_b}$. We call this operation *Combination 2*.

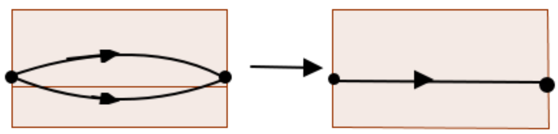

Figure 10

If is a triangle in the graph with edges a, b, c, we claim that we can re-divide the diagram formed by p_a, p_b, p_c into three rectangles p_u, p_v, p_w satisfy $R_u = \frac{R_b R_c}{R_a + R_b + R_c}$, $R_v = \frac{R_c R_a}{R_a + R_b + R_c}$, $R_w = \frac{R_a R_b}{R_a + R_b + R_c}$. Assume that $AB = x$, $AC = y$, then $AG = R_a x$, $CD = R_b y$, $EF = R_a x - R_b y$. Now, it's easy to check that when $h_u = \frac{R_b(R_b + R_c)y - R_a R_b x}{R_a + R_b + R_c}$, $v_u = \frac{R_b + R_c}{R_c} y - \frac{R_a}{R_c} x$, $h_v = \frac{R_a(R_a + R_c)x - R_a R_b y}{R_a + R_b + R_c}$, $v_v = \frac{R_a + R_c}{R_c} x - \frac{R_b}{R_c} y$, $h_u = \frac{R_a R_b}{R_a + R_b + R_c}(x + y)$, $v_u = (x + y)$, the three rectangles p_u, p_v, p_w satisfy the condition. We call this operation *Combination 3*.

Also, if there is a vertex whose degree is 3, and the edges incident to this vertex are u, v, w, we can re-divide the diagram formed by p_u, p_v, p_w into three rectangle p_a, p_b, p_c. It's easy to calculate that $R_a = R_v + R_w + \frac{R_v R_w}{R_u}$, $R_b = R_w + R_u + \frac{R_w R_u}{R_v}$, $R_c = R_u + R_v + \frac{R_u R_v}{R_w}$. We call this operation *Combination 4*.

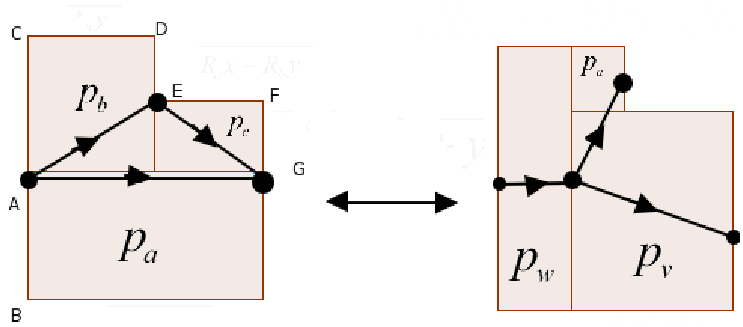

Figure 10

Now, we're going to use a theorem.

Theorem(Epifanov) Given a connected planar graph, choose any two vertices to be terminals. Then one can reduce the graph by doing Combination 1, 2, 3, 4 and deleting a vertex whose degree is 1 without deleting the terminal to an edge, which connects two terminals.

Choose the left most and the right most vertices to be terminals, then it's easy to see that there aren't any vertices whose degree is 1 except the terminals, so deleting a vertex whose degree is 1 isn't necessary. Hence, from Epifanov's Theorem we know that we can do finitely many times of Combination 1, 2, 3, 4 to make the graph contains only an edge e and two vertices. When the graph contains only an edge, there's only one rectangle remains, which is of the ratio k by the condition.

Now, we prove that if in the n^{th} combination, a rectangle p_α appears, then we can make R_α be into the set S after finitely many of operations. Use induction.

$n = 0$ is trivial since we let $S = \{r, \frac{1}{r}\}$.

If it's true that we can make R_α into S after finitely many of operations for all rectangles p_α which appear in the m^{th} combination ($0 \le m \le i$), consider the $(i+1)^{\text{th}}$ combination. If it's doing Combination 1 to rectangle p_a and p_b, then $R_\alpha = R_a + R_b$. Since R_a and R_b are both in S (because p_a and p_b already appear), we can do Operation 1 to make R_α be into S.

For the same reason, if the $(i+1)^{\text{th}}$ combination is doing Combination 2, 3, 4, we can do Operation 2, 3, 4 to make R_α be into S.

So we're done proving this by induction.

Since we can do finitely many times of combination to let the big rectangle p_e appears, we can also do finitely many times of operation to let $R_e = k$ be into S, which is what we want.

Hence the theorem is proved.

By Theorem 3, we are able to get some results.

Theorem 4 A rectangle may be dissected into squares if and only if its ratio is a positive rational number.

If a rectangle may be dissected into squares, plug r by 1 in Theorem 3, we know that the ratio of rectangle must appear in the set S obtained from finitely many operations applied to the set $\{1\}$. It's easy to check that as long as the numbers in the set are all positive and rational, then Operation 1, 2, 3, 4 can only put another positive rational number into it, so S contains only positive rational numbers. Since the ratio of rectangle must appear in S, the ratio is a positive rational number.

If a rectangle's ratio is a positive rational number, from Theorem 1, we know that it's possible to dissect it in to congruent squares, done.

And so we have answered Problem 10.

Theorem 5 If a, b are rational, s is a square-free positive integer and $a + b\sqrt{s} > 0$, then $a + b\sqrt{s}$ is tilible if and only if $a^2 > sb^2$

From Corollary 1, W.L.O.G. a and b are integers.

If $a^2 > sb^2$, let $n = a^2 - sb^2$, then from Figure 8 we tile a $1\times 2a$ rectangle successfully. So we can tile a square by rectangles of ratio $a+b\sqrt{s}$ by Theorem 1. Hence in this case $a+b\sqrt{s}$ is tilible.

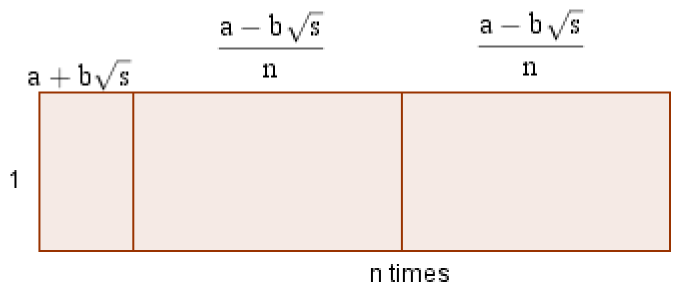

Figure 11

If $a^2 < sb^2$, set $S=\{a+b\sqrt{s}, \frac{1}{a+b\sqrt{s}}\}$, it's easy to check that Operation 1, 2, 3, 4

only put numbers of form $x+y\sqrt{s}$ into S. Moreover, we can prove that Operation 1, 2,

3 only put numbers of form $x+y\sqrt{s}$ which satisfies $x^2 < sy^2$. Indeed, define

$N(x+y\sqrt{s})$ by $N(x+y\sqrt{s}) = x^2 - sy^2$. It's easy to check that $N(uv)=N(u)N(v)$, and if

$N(x_1+y_1\sqrt{s})<0, N(x_2+y_2\sqrt{s})<0$, then

$$N((x_1+x_2)+(y_1+y_2)\sqrt{s}) = (x_1{}^2 - sy_1{}^2)+(y_1{}^2 - sy_1{}^2)+2(x_1x_2 - sy_1y_2)$$

$$< 2((\sqrt{s}y_1)(\sqrt{s}y_2) - sy_1y_2) = 0$$

So if $N(u) < 0$, $N(v) < 0$, $N(w)<0$, then $N(1/u) < 0$… (1) ,$N(u+v) < 0$…(2), $N(uvw) < 0$…(3). From (1) we know that $N(1/u) < 0$, $N(1/v) < 0$ and from (2) we

know that $N(\frac{1}{u}+\frac{1}{v})<0$, and $N(\frac{uv}{u+v})<0$ by (2). From (2) and (1) we know that

$N(\frac{1}{u+v+w})<0$, and $N(\frac{uv}{u+v+w})<0$ by (3). So is $N(\frac{vw}{u+v+w})$ and

$N(\frac{wu}{u+v+w})$. From (1) and (3) we know that $N(\frac{vw}{u})<0$, and from (2) we can get

that $N(v+w+\frac{vw}{u})<0$. So is $N(w+u+\frac{wu}{v})$ and $N(u+v+\frac{uv}{w})$. Hence, Operation

1, 2, 3, 4 only put numbers of form $x+y\sqrt{s}$ which satisfies $x^2 < sy^2$. Since $N(1) > 0$,

$a+b\sqrt{s}$ isn't tilible in this case.
In conclusion, $a+b\sqrt{s}$ is tilible if and only if $a^2 > sb^2$.

From Theorem 5, we know that $2+\sqrt{2}, 2-\sqrt{2}, 3+2\sqrt{2}, 3-2\sqrt{2}$ are tilible, and $1+\sqrt{2}, \sqrt{2}$ are not tilible. So Problem 7 and part of Problem 11 are solved, and plugging s by 2 in Theorem 5 gives the answer to Problem 12.

Theorem 6 Given three positive rational number a, b, c satisfy $ab > c$. Assume that r is the root of x^3-ax^2+bx-c, then r is tilible.

Let $q = b$, $p = c/b$-a, $t = c/pq$. It's easy to know that p, q, t is rational number. So we can tile rectangles of ratio qr, tr by rectangles of ratio r by Theorem 1. So it's possible to tile the diagram in Figure 12.

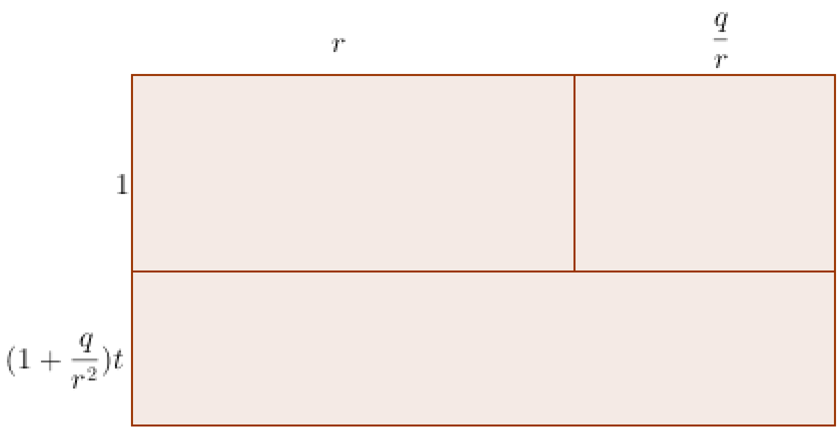

Figure 12

Since r is the root of x^3-ax^2+bx-c = x^3-$(t$+1$)px^2$+qx-pqt, we know that

$$r^3+qr=(t+1)pr^2+pqt,\ p=\frac{r^3+qr}{(t+1)r^2+qt}=\frac{r+\frac{q}{r}}{(1+\frac{q}{r^2})t+1}$$

So the big rectangle's ratio is p, and the big rectangle can form a square by Theorem 1. Hence, r is tilible.

Plugging a, b, c by 3, 3, 3 in Theorem 7, then one of the roots is $1+\sqrt[3]{2}$, since $((1+\sqrt[3]{2})-1)^3=2$. So $1+\sqrt[3]{2}$ is tilible, and we answer part of Problem 11.