The division of a segment (part 2)

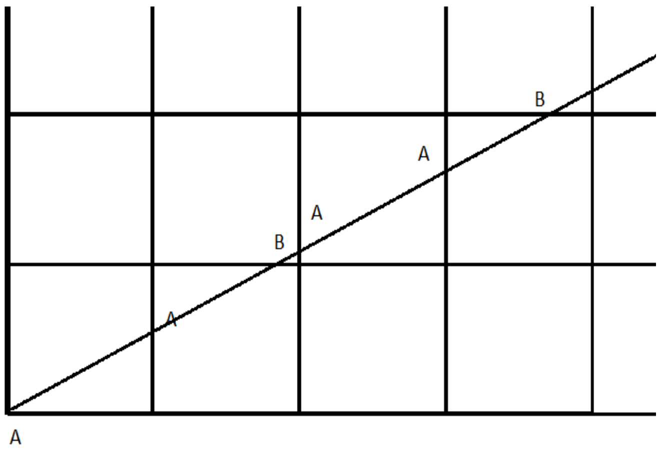
In this part we deal with problems, concerning concrete values of α.

- Let τ be the golden ratio: $\tau = \frac{1+\sqrt{5}}{2} \approx 1{,}618...$.

E1. One proposes three ways to write down a sequence of letters AABAABABAAB… Prove that all the ways give us the same result (see exact statements in points 2, 3).

Here are the ways:

(1) Draw a ray on a squared sheet of paper. It starts in some node and forms an angle arcctg τ with horizontal lines. We write letter A in the node where we start, and then write letters in intersection points of the ray with lines: letter A if the ray intersects a vertical line, and B if it intersects a horizontal line (see the picture).



(2) We write letter A and then fulfill the following steps: on every step we replace letter A with AAB, and letter B with AB. So, after three steps we get:

at first AAB,

then AABAABAB,

on the third step AABAABABAABAABABAABAB.

Prove that on each step the finite sequence is a beginning of the sequence (1).

(3) Segment [0,1] is divided as in Preliminary (in the very beginning), and let $\alpha = \tau$. Then for every double number n we write down the lengths of parts in the reverse order (from point 1 to 0). We denote the long segment by A, the short one by B.

Prove that among these sequences there are infinitely many sequences, which are beginnings of the sequence (1). In fact, it is true for $n_1 = 3 < n_2 = 8 < \dots$

E2. Choose two «intervals» from the sequence of Problem E1, n letters in each: the first interval contains letters from (k+1)-th to (k+n)-th, and the second one from (m+1)-th to (m+n)-th. Prove that the numbers of letters A in these "intervals" differ at most by 1 (so these numbers are almost equal).

E3. Find some other sequence that may be obtained analogously by the three (or at least two) ways.

- In the next set of problems we assume that $\alpha = \tau$.

P1. Find all possible values of L (for different n).

P2. What values are possible for L if n is a double number?

P3. Find all double numbers.

P4. How many double numbers are in the interval $1 < n < 10^6$? It is sufficient to give answer up to 10%.

- In the next set of problems we assume that $\alpha = \sqrt{2} \approx 1,4142...$.

T1. Find all possible values of L (for different n).
T2. What values are possible for L if n is a double number?
T3. Find all possible values of L for some other α (except $\alpha = \tau$ and $\alpha = \sqrt{2}$).

Various problems

K1. You don't know α, but you know that for any n>10, the number L takes only one of the two values. For which α it is possible?

K2. Let L takes some value A for some n. Prove that it takes the same value A for some double number n.

Let us formulate a «converse» statement: if L takes some value A for some n, than it takes the same value A for some triple number n. Is it true?

K3. **A problem for investigation.** You don't know α but you know that for any n the number of points on every segment of length ½ differs from n/2 by at most 10. What is possible to say about number α? In particular:

(a) give a few examples of such α,

(b) give some sufficient condition when the situation differs: «if α is such and such, than the statement is false»,

(c) give some condition which enables us to assert that the number of points on every segment of length ½ is between an and bn for some a< ½ <b (desirably a, b are close enough to ½).

K4. How many double and triple numbers are among n=1, 2, 3… 246 if $\alpha = 113/248$?

K5. Find some method that enables you to find in a reasonable time without a computer the quantity of double and triple numbers for α rational, $\alpha = p/q$, p<q<1.000.000 (n = 1, 2… q-1).

K6. Given the value of L if n=100: L(100) = A. Is it possible to determine whether L takes finite or infinite number of values for all n and given fixed α?

In particular, investigate the cases:

(1) A is a root of square equation with integer coefficients $A^2+qA+r=0$,
(2) A is a root of cubic equation with integer coefficients $A^3+qA^2+rA+s= 0$.

K7. It is known that number α decomposes in continued fraction and its first denominators are 3, 5, 12 (i.e. $\alpha = 1/(1+1/(3+1/(5+1/(12+\ldots))))$. Find all double numbers between 1 and 100.