Diophantine equations–1

Theorem (Gauss)**.** A positive integer d can be written as a sum of three squares if and only if d cannot be represented in the form $4^n(8m-1)$.

Introductory problems

Problem 1. Prove that the equations a) $2x^2+2xy-y^2=1$, b) $x^2-xy+y^2=2$ have no integer solutions.

Problem 2. Prove that each of the equations a) $x^2-2y^2=1$, b) $x^2-3y^2=1$, and c) $x^2-6y^2=1$ has infinitely many integer solutions.

Problem 3. Prove that the equation $x^2+1000xy+1000y^2=2001$ has infinitely many integer solutions.

Problem 4.* Fix an odd prime p. Prove that equation $x^2-py^2=-1$ has an integer solution if and only if $p\equiv 1 \pmod 4$.

Problem 5. Prove that for every integer m, the numbers of integer solutions of equations

$$x^2-xy+y^2=m \quad \text{and} \quad 3x^2+9xy+7y^2=m$$

are equal.

Problem 6. Prove that for every integer n the equation $x^2+y^2=n$ has an integer solution if and only if it has a rational solution.

Problem 7. Provide an example of a quadratic equation with integer coefficients which has a rational solution but has no integer solutions.

Problem 8. Prove that for every positive integers a and b there exist infinitely many positive integers m such that the equation $ax^2+by^2=m$ has no integer solutions.

Problem 9. Prove that for every integer m the equation $x^2+2y^2-3z^2=m$ has an integer solution.

Quadratic forms

By definition, a *quadratic form* is a homogeneous polynomial of second degree. We say that f *represents* an integer m if the equation $f=m$ has a nonzero integer solution (thus not every form represents 0.) Two quadratic forms are called *equivalent* if they represent the same set of numbers.

Problem 10. Describe all integers which are represented by forms a) x^2+y^2; b) x^2-y^2; c)* x^2+xy+y^2.

Problem 11. Prove that the quadratic forms

$$f(x,y), \quad f(x-y,y), \quad f(x,y-x), \quad f(-x,y), \quad \text{and} \quad f(x,-y)$$

are equivalent.

Problem 12. a) Prove that the forms x^2+y^2 and x^2+xy+y^2 are not equivalent.

b) Prove that the form $4x^2-6xy+5y^2$ is not equivalent to any form ax^2+by^2 with integer a and b.

Definition 1. A quadratic form is called

a) *positive definite*, if it represents only positive integers,

b) *non-negative definite*, if it represents only non-negative integers,

c) *indefinite*, if it represents the set of integers containing both positive and negative ones.

Problem 13. Provide an example of a non-negative definite form which is not positive definite.

Extended arithmetics: p-adic numbers

Theorem (Legendre)**.** Any integer is a sum of four squares.

Problem 14. Let m and n be square-free integers. Assume that the equation

$$z^2 - mx^2 - ny^2 = 0 \tag{1}$$

has a nontrivial rational solution. Prove that
a) either m or n is positive,
b) m is a quadratic residue modulo n,
c) n is a quadratic residue modulo m.

Problem 15. Reduce Metatheorem for the equations in two variables to the case of equations of the form (1).

Definition 2. An expression of the form

$$a_{-k}p^{-k} + a_{-k+1}p^{-k+1} + \ldots + a_n p^n + \ldots \qquad (k \in \mathbb{Z}, \quad a_i \in \mathbb{Z}) \tag{2}$$

is called a *p-adic number*. Such an expression is a *p-adic integer* if $k \leq 0$.

Problem 16. Let f be a polynomial with integer coefficients. Prove that the equation $f = 0$ has a solution in $\mathbb{Z}_p$ if and only if it has a solution modulo p^n for every positive integer n.

Problem 17. When is a p-adic number in the form (2) equal to 0?

Problem 18. Prove that the product of two nonzero p-adic numbers is also nonzero.

Problem 19. Prove that $\mathbb{Q} \subset \mathbb{Q}_p$ for any prime p (i.e., prove that for every pair of nonzero integers m and n there exists a p-adic number x such that $nx = m$).

Problem 20. Prove that -1 is a square in $\mathbb{Q}_p$ if and only if $p \equiv 1 \pmod 4$.

Problem 21. Find a description of all perfect squares in $\mathbb{Q}_p$.

Problem 22. Prove that for any nonzero 3-adic number m there exists a 3-adic number x such that m is equal to one of the numbers x^2, $2x^2$, $3x^2$, or $6x^2$.

Problem 23. Let p be an odd prime, and let $x_1, \ldots, x_5$ be nonzero p-adic numbers. Prove that there exist indices i and j with $1 \leq i < j \leq 5$ such that x_i/x_j is a perfect square in $\mathbb{Q}_p$.

Problem 24. Prove that for every odd prime p there exist p-adic numbers $x_1, \ldots, x_{p-1}$ such that $x_1^2 + \ldots + x_{p-1}^2 + 1 = 0$.

Problem 25. Prove that the equation $x^2 + x + 1 = 0$ has exactly two solutions in $\mathbb{Z}_7$.

Problem 26. Prove that the equation $x^2 + y^2 = -1$ has a p-adic solution for every odd prime p.

Theorem (the Hasse–Minkowski principle)**.** A quadratic equation $f = 0$ in several variables has rational solutions if and only if the equation $f = 0$ has simultaneously solutions
- in real numbers,
- in $\mathbb{Q}_p$ for every prime p.

Problem 27. Prove the Hasse–Minkowski principle for equations in one and two variables.

Definition 3. Set $(a,b)_p = 1$ if the equation $z^2 - ax^2 - by^2 = 0$ has a nonzero solution in p-adic integers; otherwise set $(a,b)_p = -1$. The value $(a,b)_p$ is the *Hilbert symbol* of the pair (a,b) with respect to the prime p.

Problem 28. Prove the following properties of the Hilbert symbol:

1) $(a,b)_p = (b,a)_p$,	2) $(a,c^2)_p = 1$,
3) $(a,-a)_p = 1, \quad (a,1-a)_p = 1$,	4) $(a,b)_p = (a,-ab)_p = (a,(1-a)b)_p$.

Problem 29. Let $(a,b)_p = 1$. Show that $(a',b)_p = (aa',b)_p$ for any a'.

Definition 4. To write down an expression for the Hilbert symbol in a compact form, we will use the *Legendre symbol* $\left(\dfrac{x}{p}\right)$ defined for any integer x and prime p. It equals to 1, -1, or 0 depending on whether x is a nonzero quadratic residue, a quadratic non-residue, or zero. For an odd prime p, one may calculate it using the formula

$$\left(\frac{x}{p}\right) = x^{\frac{p-1}{2}} \pmod p$$

Problem 30. Let p be an odd prime; let $a = p^\alpha u$, $b = p^\beta v$, where α, β, u, v are integers such that u and v are not divisible by p. Prove that

$$(a,b)_p = (-1)^{\alpha\beta\cdot\varepsilon(p)} \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha,$$

where $\varepsilon(p) = \frac{p-1}{2}$.

Problem 31. Find an explicit formula for $(a,b)_2$ for every nonzero integers a and b.

Problem 32. Prove that $(a,b)_p(a,b')_p = (a,bb')_p$ for every nonzero integers a, b, b'.

Problem 33. Prove that the equation $ax^2 + by^2 = c$ in variables x and y (with parameters a, b, and c) has a solution in p-adic numbers if and only if $(c,-ab)_p = (a,b)_p$.

Problem 34*. Let us fix a homogeneous polynomial $f = a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2$ with $n \geq 2$, where $a_1, \ldots, a_n \neq 0$. Set

$$d = a_1a_2\ldots a_n \quad \text{and} \quad \varepsilon = \prod_{i<j}(a_i,a_j)_p. \tag{3}$$

Prove that the equation $f = 0$ has a nonzero p-adic solution if and only if one of the following conditions is satisfied:

1) $n = 2$ and $-d$ is a square in $\mathbb{Q}_p$;
2) $n = 3$ and $(-1,d)_p = \varepsilon$;
3) $n = 4$ and $d \neq \alpha^2$, or $d = \alpha^2$ and $\varepsilon = (-1,-1)_p$;
4) $n \geq 5$. (i.e., if f depends on 5 or more variables, then $f = 0$ has a nonzero solution in $\mathbb{Q}_p$ for any p.)

Deduce the following problem from problem 34.

Problem 35. Fix a homogeneous polynomial $f = a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2$, where $a_1, \ldots, a_n \neq 0$, and an integer $a \neq 0$. Define d, ε by formula (3). Then the equation $f = a$ has a p-adic solution if and only if one of the following conditions is satisfied:

1) $n = 1$ and a/d is a square in $\mathbb{Q}_p$;
2) $n = 2$ and $(a, -d)_p = \varepsilon$;
3) $n = 3$ and ad is not a perfect square in $\mathbb{Q}_p$, or ad is a perfect square and $\varepsilon = (-1, -d)_p$;
4) $n \geq 4$. (i.e., if f depends on 4 or more variables, then the equation $f = a$ has a nonzero solution in $\mathbb{Q}_p$ for any p.)

Problem 36. Prove the Hasse–Minkowski principle.

Problem 37. Using problem 35 and the Hasse–Minkowski principle, show that an integer d is a sum of 3 squares in rational numbers if and only if the number d cannot be represented in the form $4^a(8b - 1)$, i.e. if $-d$ is not a perfect square in $\mathbb{Q}_2$.

Problem 38. Fix an integer n. Prove that if there exist rational numbers x, y, and z such that $x^2 + y^2 + z^2 = n$, then there also exist integers x', y', and z' such that $(x')^2 + (y')^2 + (z')^2 = n$. Deduce the Gauss theorem from this statement.

Problem 39. Deduce the Legendre theorem from the Gauss theorem.

Some properties of the Hilbert symbol (DE-2)

The goal of this section is to show that, for a pair of nonzero integers (a,b), the Hilbert symbol $(a,b)_p$ equals 1 for almost all (=all except finite number) primes p. We deduce this statement from a more general statement presented below.

Problem 40. a) Let f be a homogeneous polynomial of degree n, depending on k variables, where $k > n$. Then the number of solutions of the equivalence $f \equiv 0$ (including 0-solution!) modulo p is divisible by p (Hint: apply the little Fermat theorem and consider case $p = 2$).

b) Let f be a polynomial of degree n depending on k variables, where $k > n$. Then the number of solutions of the equivalence $f \equiv 0$ modulo p is divisible by p.

Problem 41. Deduce from the previous problem that for any integers a, b, c the equivalence $ax^2 + by^2 + cz^2 \equiv 0$ in variables x, y, z has a nonzero solution modulo p .

Problem 42. Deduce from the previous problem that, for a pair of nonzero integers (a,b) and an odd prime p, $(a,b)_p = 1$ if $a, b \not| p$. Explain why $(a,b)_p = 1$ for all primes p except a finite number.

Problem 43. Deduce from Problem 41 that the equation $ax^2 + by^2 + cz^2 + dv^2 + ew^2 = 0$ in variables x, y, z, v, w (a, b, c, d, e are parameters) has a nonzero solution in $\mathbb{Q}_p$ for any odd prime p.

Problem 44. Prove that, for any pair of nonzero integers (a,b), we have

$$\prod_p (a,b)_p = (a,b)_{-1},$$

where the product is taken over all primes p and

$$(a,b)_{-1} = \begin{cases} 1, & \text{if the equation } z^2 - ax^2 - by^2 = 0 \text{ has a real solution,} \\ -1 & \text{otherwise.} \end{cases}.$$

As a last problem of this list, we mention an "analogue" of the Chinese Remainder Theorem: it turns out that one can construct a rational number with the prescribed values of the Hilbert symbol.

Problem 45. Fix a finite set of nonzero integers a_i and for every prime p define the values $\varepsilon_{i,p} = \pm 1$. Show that the system of equations

$$(a_i, x)_p = \varepsilon_{i,p}, \quad \forall i, \ \forall p,$$

has a solution if and only if

a) almost all (=all except finite number) $\varepsilon_{i,p} = 1$,

b) for any prime p, there exists a nonzero p-adic number x_p such that

$$(a_i, x_p) = \varepsilon_{i,p}.$$

Two variables: maps of quadratic forms (DE-4)

In this section we study the equation

$$E_m: \quad ax^2 + bxy + cy^2 = m \tag{4}$$

depending on integer variables x, y, where a, b, c, m are integer parameters.

Problem 46 (Superproblem)**.** Prove that if the equation E_m has a solution for some positive m, has a solution for some negative m, has no non-trivial solutions for $m = 0$, then for every m either E_m has no solutions, or E_m has infinitely many solutions.

Problem 47 (Superproblem)**.** Is it true that if the equation E_m has solutions for

$$m = \pm 1, \ \pm 2, \ \pm 3,$$

then in this case E_m has solutions for any integer m?

Problem 48 (Superproblem)**.** Prove that if the equations E_1, E_2, E_3, E_5 have integer solutions, then the equation E_m has an integer solution for some $m < 0$.

Drawing a map

Problem 49. Prove that, if $\{w_1, w_2\}$ is a basis of $\mathbb{Z}^2$, then pairs

$$\{w_2, w_1\}, \{w_1 - w_2, w_2\}, \{w_1 + w_2, w_2\}, \{-w_1, w_2\} \tag{5}$$

are also bases of $\mathbb{Z}^2$.

Problem 50. Show that, using transformations (5), it is possible to transform any basis to any other one.

Problem 51. Show that a quadratic form can have the same representations in several different bases.

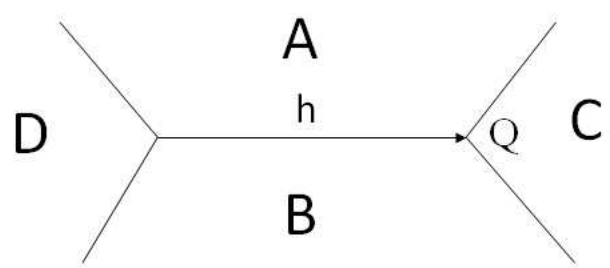
Problem 52. Find a quadratic form which has different representations in any two different bases of $\mathbb{Z}^2$.

Excercise 1. Write down all the extensions of a basis $\{w_1, w_2\}$. Write down all the specializations of a superbasis $\{\pm w_1, \ \pm w_2, \ \pm(w_1 + w_2)\}$.

Excercise 2. Draw (oriented) maps of the following quadratic forms:

$$f_1 = 3x^2 + 9xy + 7y^2, \quad f_2 = x^2 - 2y^2, \quad f_3 = x^2 - 3y^2.$$

In two problems below, the values A, B, C, D, and h are related to the following picture.



Problem 53. Show that A, B, C, D, and h satisfy

$$C = A + B + h, \qquad D = A + B - h.$$

Problem 54. Assume that A, B, C are positive and the edge h goes from C to D. Show that in this case D is also positive and that the arrows on two other edges which are incident to Q go out of Q.

Problem 55. Show that the graph determined by the points-superbases and edges-bases is a tree, i.e., it has no cycles.

Problem 56. Let Q be a unique well of a positive definite quadratic form f, and p, q, r be integers written in the regions adjacent to Q. Show that the number in any other region of a map related to f is strictly greater than $\max(p, q, r)$.

Problem 57. Prove that every positive definite form has a well.

Problem 58. a) Prove that a positive definite form has not more than two wells.
b) Find a positive definite form with two wells.

Problem 59. Provide an algorithm which solves the equation $ax^2 + bxy + cy^2 = m$ (a, b, c, m are parameters, x, y, z are variables), under the assumption that $ax^2 + bxy + cz^2$ is positive definite.

Problem 60 (Classification of positive definite quadratic forms)**.**

a) Show that any positive definite quadratic form is equivalent to the form

$$(p+q)x^2 + 2qxy + (q+r)y^2 \tag{6}$$

for some non-negative numbers p, q, r.

b) Show that the quadratic forms corresponding to

$$(p_1, q_1, r_1) \text{ and } (p_2, q_2, r_2)$$

are equivalent if and only if these triples coincide as multisets.

c) Find out which triples (p, q, r) determine an integer quadratic form.

d) Find out which triples (p, q, r) determine a positive definite quadratic form.

Part 3: Little Methuselah form

The goal of this section is to prove the following theorem.

Theorem. (Conway) Little Methuselah form $x^2 + 2y^2 + yz + 4z^2$ represents all the integers from 1 to 30. Any other positive definite form $f(x, y, z)$ which represent all the integers from 1 to 30 is linearly equivalent to the little Methuselah form.

To prove the Conway theorem, we try to develop a theory of positive definite quadratic forms in three variables. First we revisit the theory of quadratic forms in two variables.

Let $f(x, y) = ax^2 + bxy + cy^2$ be a quadratic form. We assign the following 2x2 and 3x3 tables

$$F := \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}, \quad \hat{F} := \begin{pmatrix} a & \frac{b}{2} & -(a + \frac{b}{2}) \\ \frac{b}{2} & c & -(c + \frac{b}{2}) \\ -(a + \frac{b}{2}) & -(c + \frac{b}{2}) & (a + b + c) \end{pmatrix}$$

to such a form. It is easy to see that f may be uniquely recovered from the tables F and $\hat{F}$.

Problem 61. Prove that

$$f(x, y) = -\frac{b}{2}(x - y)^2 + (a + \frac{b}{2})x^2 + (c + \frac{b}{2})y^2. \tag{7}$$

.

Problem 62. Prove that the tables

$$\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}, \quad \begin{pmatrix} c & \frac{b}{2} \\ \frac{b}{2} & a \end{pmatrix}, \quad \begin{pmatrix} a & -\frac{b}{2} \\ -\frac{b}{2} & c \end{pmatrix} \tag{8}$$

determine equivalent quadratic forms.

Problem 63. Prove that the quadratic forms corresponding to the tables

$$\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}, \quad \begin{pmatrix} a & -(a + \frac{b}{2}) \\ -(a + \frac{b}{2}) & a + b + c \end{pmatrix}, \quad \begin{pmatrix} c & -(c + \frac{b}{2}) \\ -(c + \frac{b}{2}) & a + b + c \end{pmatrix} \tag{9}$$

are equivalent (note that tables (9) can be obtained from $\hat{F}$ by a choice of 2 rows and corresponding 2 columns).

Below, we identify a quadratic form f with its tables F and $\hat{F}$.

Problem 64. Using (8) and (9), show that any positive definite quadratic form is equivalent to a form

$$\begin{pmatrix} a' & \frac{b'}{2} \\ \frac{b'}{2} & c' \end{pmatrix},$$

for which $0 \le -b' \le a' \le c'$. Note that under this restriction, the right hand side of (7) is a sum of 3 non-negative numbers.

Problem 64 is an analogue of Problem 60. We wish to prove the analogue of Problem 64 for quadratic forms in three variables. We will use the same scheme but we need more notation. Fix a quadratic form

$$f(x,y,z) = a_{xx}x^2 + a_{yy}y^2 + a_{zz}z^2 + a_{xy}xy + a_{yz}yz + a_{xz}xz.$$

We identify the form f with the following 3x3 and 4x4 tables:

$$F := \begin{pmatrix} a_{xx} & \frac{a_{xy}}{2} & \frac{a_{xz}}{2} \\ \frac{a_{xy}}{2} & a_{yy} & \frac{a_{yz}}{2} \\ \frac{a_{xz}}{2} & \frac{a_{yz}}{2} & a_{zz} \end{pmatrix},$$

$$\hat{F} := \begin{pmatrix} a_{xx} & \frac{a_{xy}}{2} & \frac{a_{xz}}{2} & -(a_{xx} + \frac{a_{xy}}{2} + \frac{a_{xz}}{2}) \\ \frac{a_{xy}}{2} & a_{yy} & \frac{a_{yz}}{2} & -(a_{yy} + \frac{a_{xy}}{2} + \frac{a_{yz}}{2}) \\ \frac{a_{xz}}{2} & \frac{a_{yz}}{2} & a_{zz} & -(a_{zz} + \frac{a_{xz}}{2} + \frac{a_{yz}}{2}) \\ -(a_{xx} + \frac{a_{xy}}{2} + \frac{a_{xz}}{2}) & -(a_{yy} + \frac{a_{xy}}{2} + \frac{a_{yz}}{2}) & -(a_{zz} + \frac{a_{xz}}{2} + \frac{a_{yz}}{2}) & \begin{matrix} a_{xx} + a_{yy} + a_{zz} + \\ + \frac{a_{xy}}{2} + \frac{a_{yz}}{2} + \frac{a_{xz}}{2} \end{matrix} \end{pmatrix}.$$

Problem 65. Prove that

$$\begin{aligned} f(x,y,z) = &-\frac{a_{xy}}{2}(x-y)^2 - \frac{a_{xz}}{2}(x-z)^2 - \frac{a_{yz}}{2}(y-z)^2 + \\ &+ (a_{xx} + \frac{a_{xy}}{2} + \frac{a_{xz}}{2})x^2 + (a_{yy} + \frac{a_{xy}}{2} + \frac{a_{yz}}{2})y^2 + (a_{zz} + \frac{a_{xz}}{2} + \frac{a_{yz}}{2})z^2. \end{aligned} \tag{10}$$

Problem 66. Prove that the quadratic forms

$$\begin{pmatrix} a_{xx} & \frac{a_{xy}}{2} & \frac{a_{xz}}{2} \\ \frac{a_{xy}}{2} & a_{yy} & \frac{a_{yz}}{2} \\ \frac{a_{xz}}{2} & \frac{a_{yz}}{2} & a_{zz} \end{pmatrix}, \quad \begin{pmatrix} a_{xx} & \frac{a_{xy}}{2} & -(a_{xx} + \frac{a_{xy}}{2} + \frac{a_{xz}}{2}) \\ \frac{a_{xy}}{2} & a_{yy} & -(a_{yy} + \frac{a_{xy}}{2} + \frac{a_{yz}}{2}) \\ -(a_{xx} + \frac{a_{xy}}{2} + \frac{a_{xz}}{2}) & -(a_{yy} + \frac{a_{xy}}{2} + \frac{a_{yz}}{2}) & \begin{matrix} a_{xx} + a_{yy} + a_{zz} + \\ + \frac{a_{xy}}{2} + \frac{a_{yz}}{2} + \frac{a_{xz}}{2} \end{matrix} \end{pmatrix}, \tag{11}$$

$$\begin{pmatrix} a_{xx} & \frac{a_{xz}}{2} & -(a_{xx} + \frac{a_{xy}}{2} + \frac{a_{xz}}{2}) \\ \frac{a_{xz}}{2} & a_{zz} & -(a_{zz} + \frac{a_{xz}}{2} + \frac{a_{yz}}{2}) \\ -(a_{xx} + \frac{a_{xy}}{2} + \frac{a_{xz}}{2}) & -(a_{zz} + \frac{a_{xz}}{2} + \frac{a_{yz}}{2}) & \begin{matrix} a_{xx} + a_{yy} + a_{zz} + \\ + \frac{a_{xy}}{2} + \frac{a_{yz}}{2} + \frac{a_{xz}}{2} \end{matrix} \end{pmatrix}, \tag{12}$$

$$\begin{pmatrix} a_{yy} & \frac{a_{yz}}{2} & -(a_{yy} + \frac{a_{xy}}{2} + \frac{a_{yz}}{2}) \\ \frac{a_{yz}}{2} & a_{zz} & -(a_{zz} + \frac{a_{xz}}{2} + \frac{a_{yz}}{2}) \\ -(a_{yy} + \frac{a_{xy}}{2} + \frac{a_{yz}}{2}) & -(a_{zz} + \frac{a_{xz}}{2} + \frac{a_{yz}}{2}) & \begin{matrix} a_{xx} + a_{yy} + a_{zz} + \\ + \frac{a_{xy}}{2} + \frac{a_{yz}}{2} + \frac{a_{xz}}{2} \end{matrix} \end{pmatrix} \tag{13}$$

are equivalent (note that tables (13) can be obtained from $\hat{F}$ by a choice of 3 rows and 3 corresponding columns).

Problem 67. Using (13), show that a positive definite form f is equivalent to a form

$$\begin{pmatrix} a'_{xx} & \frac{a'_{xy}}{2} & \frac{a'_{xz}}{2} \\ \frac{a'_{xy}}{2} & a'_{yy} & \frac{a'_{yz}}{2} \\ \frac{a'_{xz}}{2} & \frac{a'_{yz}}{2} & a'_{zz} \end{pmatrix},$$

for which

$$0 < a'_{xx} \le a'_{yy} \le a'_{zz},$$

$$|a'_{xy}|, |a'_{xz}| \le |a'_{xx}|, |a'_{yz}| \le |a'_{zz}|.$$

Problem 68. Using Problem 67, show that every positive definite quadratic form $f(x, y, z)$ is equivalent to

$$\hat{F} := \begin{pmatrix} a'_{xx} & \frac{a'_{xy}}{2} & \frac{a'_{xz}}{2} & -(a'_{xx} + \frac{a'_{xy}}{2} + \frac{a'_{xz}}{2}) \\ \frac{a'_{xy}}{2} & a'_{yy} & \frac{a'_{yz}}{2} & -(a'_{yy} + \frac{a'_{xy}}{2} + \frac{a'_{yz}}{2}) \\ \frac{a'_{xz}}{2} & \frac{a'_{yz}}{2} & a'_{zz} & -(a'_{zz} + \frac{a'_{xz}}{2} + \frac{a'_{yz}}{2}) \\ -(a'_{xx} + \frac{a'_{xy}}{2} + \frac{a'_{xz}}{2}) & -(a'_{yy} + \frac{a'_{xy}}{2} + \frac{a'_{yz}}{2}) & -(a'_{zz} + \frac{a'_{xz}}{2} + \frac{a'_{yz}}{2}) & \begin{matrix} a'_{xx} + a'_{yy} + a'_{zz} + \\ + \frac{a'_{xy}}{2} + \frac{a'_{yz}}{2} + \frac{a'_{xz}}{2} \end{matrix} \end{pmatrix} \tag{14}$$

such that

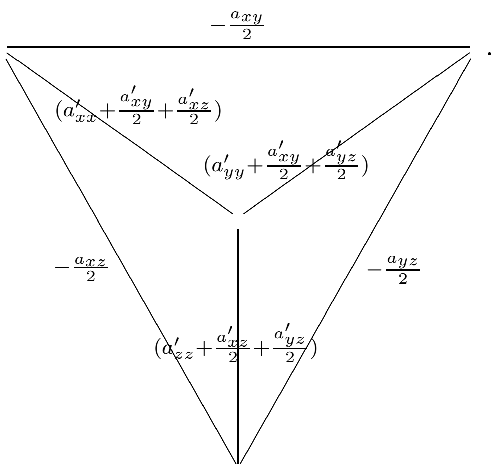
$$a'_{xy},\ a'_{yz},\ a'_{xz} \le 0, \tag{15}$$

$$(a'_{xx} + \frac{a'_{xy}}{2} + \frac{a'_{xz}}{2}) \ge 0,\ (a'_{yy} + \frac{a'_{xy}}{2} + \frac{a'_{yz}}{2}) \ge 0,\ (a'_{zz} + \frac{a'_{xz}}{2} + \frac{a'_{yz}}{2}) \ge 0. \tag{16}$$

Note under these conditions the right hand side of (10) is a sum of squares.

To every quadratic form (14) satisfying conditions (16), we assign a graph with 4 vertices as it is shown on Figure below.

$\mathcal{F} =$



If the number on some edge is 0, then we delete this edge.

Problem 69. Prove that if the graph $\mathcal{F}$ has all possible edges, then f does not represent 1.

Problem 70. Prove that if some vertex of $\mathcal{F}$ is incident to strictly less than 2 edges, then f is equivalent to a form

$$ax^2 + g(y, z) \tag{17}$$

for some positive integer a and a positive definite quadratic form g in 2 variables.

Problem 71. Prove that if a quadratic form (17) represents all the integers from 1 to 30, then (17) is equivalent to the little Methuselah form.

We say that f is *indecomposable* if any vertex of $\mathcal{F}$ is incident to at least 2 edges.

Problem 72. Describe the graphs of all indecomposable quadratic forms $f(x, y, z)$ which represent

a) 1;
b) 1, 2;
c) 1, 2, 3, 5.

Problem 73. Finish the proof of the Conway theorem.