Planar arrangements of lines

Solutions of problems before intermediate finish

Problem 1. The answer is: $n+1 \leqslant f \leqslant \frac{n(n+1)}{2}+1$. The both inequalities can be proved by induction on n. The base of induction $n=1$, $f=2$ is evident. Let us add one line. If it has x intersection points with the old lines, then the number of regions increases by $x+1$ after adding this line. Hence, the nth line increases the number of regions by at least 1 and at most n.

Problem 2. The answer is:

n=1, $f=2$,

n=2, $f \in \{3,4\}$,

n=3, $f \in \{4,\ 6,7\}$,

n=4, $f \in \{5,\ 8,\dots,11\}$,

n=5, $f \in \{6,\ 10,\ 12,\dots,16\}$,

n=6, $f \in \{7,\ 12,\ 15,\dots,22\}$,

n=7, $f \in \{8,\ 14,\ 18,\dots,29\}$.

The impossibility of the other values can be deduced from problems 1 and 3(a)(b).

Problem 3. (a) Let us consider p parallel lines of the arrangement and then add the other lines one by one. Each added line has at least p intersection points with the preceding lines, hence, the number of regions increases by at least $p+1$. In total, if we add $n-p$ lines, we obtain $f \geqslant (p+1)(n-p+1)$.

(b) Let us consider q concurrent lines and then add all the remaining lines one by one. Every new line has at least $q-1$ intersection points with the preceding lines, hence the number of regions increases by at least q after adding this line. If we add $n-q$ lines, we obtain $f \geqslant q(n-p+2)$.

(c) Let us consider p parallel lines, then fix a point on one of these lines and pass $n-p$ other lines through this point one by one. Each new line increases the number of regions by p, and finally we obtain $(p+1)(n-p)$ regions.

Let us consider q concurrent lines. All the other $n-q$ lines will be parallel to one chosen line of these q lines. Pass them one by one. Every new line increases the number of regions by q, so we obtain $q(n-q+2)$ regions.

(d) *Solution 1.* Let us add the lines one by one. If the line has x intersection points with the preceding lines, then the number of regions increases by $x+1$. On the other hand, each of these x intersection points either has multiplicity 2, or one plus its previous multiplicity. Hence the sum $\sum_{i\geqslant 2}(i-1)r_i$ increases by x after adding this line.

Solution 2. Let us intersect the given arrangement with the disk of a sufficiently big radius (such that the disc contains all the intersection points of lines). Consider the following graph: its vertices are the intersection points of lines with each other and with the boundary of the disc, its edges are the line and circle segments which do not contain intersection points except its ends. The number of vertices of this graph equals $v=2n+\sum_{i\geqslant 2} r_i$, the number of edges equals $e = 3n+\sum_{i\geqslant 2} ir_i$. The number of regions f of the plane is less by one than the number of the regions of the disc divided by the edges of the graph. Using the Euler formula for the graphs, we obtain

$$f = n + 1 + \sum_{i\geqslant 2}(i-1)r_i.$$

Problem 4. (a) If all the lines are parallel, then $f = p+1$. If not all the lines are parallel, then $1 \leqslant p \leqslant n-1$. Using problem 3, we obtain $f \geqslant (p+1)(n-p+1)$. The expression $(p+1)(n-p+1)$ is the quadratic trinomial in p with the negative leading coefficient. For $p = 1$ and $p = n-1$, the value of the trinomial $(p+1)(n-p+1)$ is $2n$. Hence, for $1 \leqslant p \leqslant n-1$ the value of the trinomial $(p+1)(n-p+1)$ is not less than $2n$.

(b) If all or all except one lines are parallel, then $f = n+1$ or $f = 2n$. Otherwise, using problem 3(a), we have $f \geqslant (p+1)(n-p+1)$. The expression $(p+1)(n-p+1)$ is a quadratic trinomial with respect to p with the negative leading coefficient. For $p = 2$ and $p = n-2$ the value of the trinomial $(p+1)(n-p+1)$ equals $3n-3$. Hence, for $2 \leqslant p \leqslant n-2$ the value of the trinomial $(p+1)(n-p+1)$ is not less than $3n-3$. Consequently, for $2 \leqslant p \leqslant n-2$ we have $f \geqslant 3n-3$. If $p = 1$, then, using problem 3(b), we have $f \geqslant q(n-q+2)$. If $q = n$, then $f = 2n$. Consider the quadratic trinomial $q(n-q+2)$. For $3 \leqslant q \leqslant n-1$ we have $f \geqslant q(n-q+2) \geqslant 3n-3$. The remaining case is $p = 1$, $q = 2$, where the number of regions is

$$1 + \frac{n(n+1)}{2} \geqslant 3n-3 \quad \Leftrightarrow \quad n^2 - 5n + 8 \geqslant 0.$$

(c) If $p \geqslant n-2$ or $q \geqslant n-1$, then $f \leqslant 3n-2$. If $3 \leqslant p \leqslant n-3$ or $4 \leqslant q \leqslant n-2$, then by problem 3 we have $f \geqslant 4n-8$. The remaining case is $p \leqslant 2$ and $q \leqslant 3$. By problem 3 we have $f = n+1+r_2+2r_3$. Each line of the arrangement intersects at least $n-2$ other lines. Hence, the number of pairs of intersecting lines equals r_2+3r_3, which is not less than $\frac{n(n-2)}{2}$. Consequently,

$$f \geqslant n+1+\frac{2}{3}(r_2+3r_3) \geqslant n+1+\frac{n^2-2n}{3} \geqslant 4n-8$$

for $n \geqslant 8$.

Problem 5. (a) Consider p parallel lines in the arrangement of n lines and add the remaining lines one by one. The line number i, $1 \leqslant i \leqslant n-p$, intersects the preceding lines in at most $p+i-1$ points, hence the number of regions increases by at most $p+i$. If we add $n-p$ lines, we obtain

$$f \leqslant (p+1)(n-p+1) + C_{n-p}^2.$$

The bound is reached for the arrangement containing p parallel lines and $n-p$ lines in general position (i.e. no two are parallel and no three are concurrent), such that they are also in general position with the p parallel lines.

(b) Consider q concurrent lines in the arrangement of n lines and add the remaining lines one by one. The line number i, $1 \leqslant i \leqslant n-q$, intersects the preceding ones in at most $q+i-1$ points, hence the number of regions increases by at most $q+i$. If we add $n-q$ lines, we obtain

$$f \leqslant q(n-p+2) + C_{n-q+1}^2 = 1 + C_{n+1}^2 - C_{q-1}^2.$$

The bound is reached for the arrangements of q concurrent lines and $n-q$ lines in general position such that they are also in general position with the q concurrent lines.

Problem 6. Let us take $n-p$ lines in general position and choose some t intersection points of these lines, where $t \leqslant \min\{p, C_{n-p}^2\}$. Now add p parallel lines in such a way that they pass through t chosen points but do not pass through the other intersection points of the $n-p$ lines. Also, the p added lines should not be parallel to any of the first $n-p$ lines. Then it is easy to check that this arrangement has at most p parallel lines, and that the number of regions equals

$$f = (p+1)(n-p+1) + C_{n-p}^2 - t.$$

Problem 7. For given n and p, the minimal number of regions of the arrangement equals $(p+1)(n-p+1)$. To reach it, we can take the arrangement containing $n-p+1$ concurrent lines and $p-1$ parallel lines such that the parallel lines are also parallel to one chosen line of the first $n-p+1$ ones. Now note that

$$(p+1)(n-p+1) < a(n,p) \;\Leftrightarrow\; C_{n-p}^2 > p \;\Leftrightarrow\; n > p + \frac{1}{2} + \sqrt{2p+\frac{1}{4}}.$$

The second inequality is also equivalent to $n \geqslant p + \frac{1}{2} + \sqrt{2p+\frac{9}{4}}$ (if we write $C_{n-p}^2 \geqslant p+1$).

Problem 8. (a)

$$L(n) = \max\{k \geqslant 1 \mid b(n, n-k+1) \leqslant a(n, n-k) - 2\} \text{ for } n \geqslant 3.$$

$$b(n,n-k+1) \leqslant a(n,n-k) - 2 \quad\Leftrightarrow\quad \min\left\{n-k,\ \frac{k(k-1)}{2}\right\} \leqslant n-k-2 \quad\Leftrightarrow$$

$$\Leftrightarrow\quad n \geqslant \frac{k^2+k}{2} + 2 \quad\Leftrightarrow\quad k \leqslant \sqrt{2n - \frac{15}{4}} - \frac{1}{2}.$$

It follows that $L(n) = \left[\sqrt{2n-\frac{15}{4}} - \frac{1}{2}\right]$.

(b) Since

$$n \geqslant \frac{L^2(n)+L(n)}{2} + 2, \text{ then } \min\left\{n-j,\ \frac{j(j-1)}{2}\right\} = \frac{j(j-1)}{2} \text{ for } 1 \leqslant j \leqslant L(n).$$

Hence, $a(n, n-j) = (n-j+1)(j+1)$, and the jth gap contains $n - \frac{j(j+1)}{2} - 1$ integers.

Problem 9. If $j \leqslant p \leqslant n-j$ or $j+1 \leqslant q \leqslant n-j+1$, then by problem 3 we have $f \geqslant (n-j+1)(j+1) = a(n, n-j)$. If $p \geqslant n-j+1$, then, as follows from problem 5,

$$f \leqslant (p+1)(n-p+1) + C_{n-p}^2 \leqslant b(n, n-j+1).$$

If $q \geqslant n-j+2$, then by problem 5 we have

$$f \leqslant q(n-q+2) + C_{n-q}^2 + 1 \leqslant b(n, n-j+1).$$

Problem 10. The number of pairs of intersecting lines equals $\sum_{i=2}^{q} \frac{i(i-1)}{2} r_i$ which is not less than $\frac{n(n-p)}{2}$, since every line intersects at least $n-p$ lines.

Problem 11. (a) Using problems 3 and 10, we obtain that

$$f - (n+1) = \sum_{i\geqslant 2} (i-1) r_i \geqslant \sum_{i\geqslant 2}^{q} \frac{i(i-1)}{q} r_i \geqslant \frac{n(n-p)}{q}.$$

(b) Suppose the contrary, then by problem 9 we have $p \leqslant j-1$ and $q \leqslant j$, where j is the number of the gap. Note that $n-p \geqslant n-j+1$ and $n+1 \geqslant n-j+1$. It contradicts with (a):

$$f \geqslant n+1+\frac{n(n-p)}{q} \geqslant (n-j+1)(\frac{n}{q}+1) \geqslant (n-j+1)(j+1) = a(n, n-j).$$

Problem 12. Let us intersect the given arrangement with a disc of a sufficiently big radius, such that it contains all the intersection points. We obtain a graph: its vertices are the intersection points of the lines of the arrangement and the intersection points of the lines with the circle, its edges are the line and circle segments, which do not contain intersection points except their ends. The number of vertices v and the number of edges e equal $2n+\sum_{i\geqslant 2} r_i$ and $3n+\sum_{i\geqslant 2} ir_i$, respectively. The interior of the disc contains f regions, where

$$f = n+1+\sum_{i\geqslant 2}(i-1)r_i.$$

The sum of the numbers of bounding edges for all the regions in the disc equals

$$2e-2n = 4n+2\sum_{i\geqslant 2} ir_i.$$

Since $p < n$, every region in the disc is bounded by at least three edges of the graph. So we have $2e-2n \geqslant 3f$, which gives the required inequality.

Problem 13. (a) Let $a = \frac{2}{q+3}$ and $b = \frac{q-1}{q+3}$. Consider the following quadratic trinomial in the variable i:

$$a(i^2-i)+b(3-i)-(i-1) = a(i-2)(i-q) \leqslant 0 \quad \text{for} \quad 2 \leqslant i \leqslant q.$$

Let us multiply its corresponding values by r_i and sum up for all i, $2 \leqslant i \leqslant q$:

$$0 \geqslant a\sum_{i=2}^{q} i(i-1)r_i + b\sum_{i=2}^{q}(3-i)r_i - \sum_{i=2}^{q}(i-1)r_i \geqslant an(n-p)+b(3-n)-(f-n-1),$$

where the last inequality is obtained from problems 10, 12, and 3. Consequently,

$$f \geqslant 2\frac{n(n-p)}{q+3} + (n+1+\frac{q-1}{q+3}(3-n)) \geqslant 2\frac{n(n-p)}{q+3}.$$

(b) Suppose the contrary, then by problem 9 we have $p \leqslant j-1$ and $q \leqslant j$. Using that $n-p \geqslant n-j+1$, we get a contradiction with (a):

$$f \geqslant 2\frac{n(n-p)}{q+3} \geqslant (n-j+1)\frac{2n}{q+3} \geqslant (n-j+1)(j+1) = a(n, n-j),$$

because

$$\frac{(q+3)(j+1)}{2} \leqslant \frac{j^2+4j+3}{2} \leqslant \frac{1}{2}L^2(n) < n.$$

Problem 14. (a) The intersection point determines two diagonals passing through it and the ends of these diagonals, which form a four-tuple of vertices of the given n-gon.

Conversely, every four-tuple of distinct vertices of the n-gon determines one intersection point of diagonals. Hence, the number of intersection points is C_n^4.

(b) *Solution 1.* Let us pass the diagonals one by one. If the added diagonal has x intersection points with the previous diagonals, then the number of regions of the interior of the n-gon is increased by $x+1$. We can treat it as every diagonal and every intersection point increases the number of regions by 1. Hence, the required number of regions equals $1+C_n^4+\frac{n(n-3)}{2}$, because the number of intersection points was found in (a), and the number of diagonals equals $\frac{n(n-3)}{2}$.

Solution 2. Let us consider a graph: its vertices are the vertices of the n-gon and the intersection points of the diagonals, its edges are the diagonal segments and the sides of the n-gon. In this graph, the number of vertices equals $v=n+C_n^4$, and the number of edges equals $e=2C_n^4+\frac{n(n-1)}{2}$. By the Euler formula, the number of regions equals $1+e-v=1+C_n^4+\frac{n(n-3)}{2}$.

Problem 15. See problem 23.

Problem 16. Let us draw the lines each of which passes through exactly two of given points. These m lines divide the plane into at most $1+m+C_m^2$ regions. Consider all the points of the given set such that for each of these point, no line passing through exactly two points of the given set, passes through this point. It turns out that every region formed by the arrangement of m lines contains at most one such point (prove it yourself). Also find yourself m regions not containing such points. In total, m lines contain at most $2m$ points of the initial set. Hence $n \leqslant 2m+1+C_m^2=C_{m+2}^2$.

The solutions of the problems after intermediate finish

Problem 17. For the parallel planes α_1 and α_2, the central projection is just a dilatation, which maps lines to lines and regions to regions. Now suppose that α_1 and α_2 intersect, and let l_1 and l_2 be the intersection lines of the planes α_1 and α_2 with planes passing through the point O parallel to planes α_2 and α_1, respectively. If a line l does not coincide with l_1, then its image is a line for $l \| l_1$, and is a pointed line if $l \nparallel l_1$ (the line with one point thrown out, namely the point of intersection with l_2). If a region intersects l_1, then its image consists of two unbounded regions. If a non-bounded region is incident to two non parallel rays, then its image is incident to l_2.

Problem 18. It is one-to-one by definition. The image and the pre-image will be the lines l_2 and l_1 together with their infinite points. Recall that l_1 and l_2 are the lines of intersection of the planes α_1 and α_2 with the planes passing through O parallel to α_2 and α_1, respectively.

Problem 19. Every two lines have exactly one common points on the plane considered with its infinite points (if these lines are parallel or one of the lines is infinite, then this point is infinite). The number of pairs of lines is C_n^2. If an intersection point belongs to i points, it is an intersection point for C_i^2 pairs of lines, which gives the required.

Problem 20. First we suppose that not all the lines of the arrangement of n lines are concurrent, and define the "region" in the projective plane in this case. If one of the lines is infinite, then the regions coincide with the regions of the usual plane divided by $n-1$ remaining lines. If the arrangement does not contain the infinite line, then the following objects will be the regions:

- a bounded region of the plane,

- a non-bounded region of the plane incident to two parallel rays,
- a pair of non-bounded regions of the plane, the first one incident to two non-parallel rays l_1 and l_2, the second one to non-parallel rays l_3 and l_4, such that $l_1 \| l_3$ and $l_2 \| l_4$. The infinite points corresponding to the directions which are between l_1 and l_2 also belong to this region, they "glue up" two its infinite parts.

If all the lines of the arrangement are concurrent, "regions" are defined in a similar way. Two points belong to the same region if and only if they are connected by a polygonal line not intersecting the lines of the arrangement. Here the ends of the polygonal line are the two given points, and the other vertices can be infinite. Its segments incident to an infinite vertex are rays which are parallel to the direction of this infinite point. After this re-definition, the polygonal line is mapped to a polygonal line via a central projection. Consequently, two points belong to one region if and only if their images via the central projection belong to one region. So the correspondence between regions is one-to-one.

Problem 21. (a) *Solution 1.* Let us map one line of the arrangement to the infinite line via a central projection. The number of regions and the values t_i do not change, but now the number of regions of the projective plane coincides with the number of regions f of the usual plane, divided by the arrangement of $n-1$ remaining lines. By problem 3, we have $f = n + \sum_{i\geqslant 2}(i-1)r_i$. Now note that

$$n - 1 = \sum_{i\geqslant 2}(i-1)(t_i - r_i),$$

since the infinite line contains $t_i - r_i$ intersection points, which belong to $i-1$ lines of the arrangement, and plus the infinite point.

Solution 2. Induction on n. The base $n = 1$ is evident. The step follows from the fact that each added line increases the number of regions by the number of the points of intersection with the previous lines.

(b) Induction on n. The base $n = m$, $f = m$ is evident. The added line number j, $1 \leqslant j \leqslant n-m$, has at least m and at most $m+j-1$ points of intersection with the previous lines, since every two lines intersect in the projective plane. The required inequality follows from the fact that each added line increases the number of regions by the number of points of intersection with the previous lines.

(c) Consider the graph in the projective plane: its vertices are the intersection points of lines, the edges are the line segments. We allow edges to pass through infinite points, such an edge consists of two rays belonging to the same line together with the infinite point of this direction which glues up the two rays. The number of vertices v and the number of edges e of this graph equal $\sum_{i\geqslant 2} r_i$ and $\sum_{i\geqslant 2} ir_i$, respectively. Since $m < n$, each region is bounded by at least three edges of the graph, hence

$$3f \leqslant 2e \;\Leftrightarrow\; 3 + 3\sum_{i\geqslant 2}(i-1)t_i \leqslant 2\sum_{i\geqslant 2} ir_i \;\Leftrightarrow\; \sum_{i\geqslant 2}(3-i)t_i \geqslant 3.$$

(d) Let $a = \frac{2}{M+3}$ and $b = \frac{M-1}{M+3}$. Consider the following quadratic trinomial in variable i:

$$a(i^2 - i) + b(3-i) - (i-1) = a(i-2)(i-M) \leqslant 0 \quad \text{if} \quad 2 \leqslant i \leqslant m.$$

Multiply the corresponding values of this trinomial by t_i and sum up for all i, $2 \leqslant i \leqslant m$:

$$0 \geqslant a\sum_{i=2}^{m} i(i-1)t_i + b\sum_{i=2}^{m}(3-i)t_i - \sum_{i=2}^{m}(i-1)t_i \geqslant an(n-1) + 3b - (f-1),$$

where the last inequality follows from problems 19, 21(a) and (c). Consequently,

$$f \geqslant 2\frac{n(n-1)}{M+3} + \left(1 + \frac{3(M-1)}{M+3}\right) = 2\frac{n(n-1)+2M}{M+3}.$$

Problem 22*. Let us choose two points P and Q incident to m lines each, and denote by N the set of lines of the initial arrangement passing through at least one of P and Q. Two cases are possible:

(i) the line PQ does not belong to the initial arrangement. Then N contains $2m$ lines and divides the projective plane into m^2+2m-1 regions (if we do not consider other lines).

(ii) the line PQ belongs to the initial arrangement. Then N contains $2m-1$ lines and divides the projective plane into m^2 regions (if we do not consider other lines).

In both cases every remaining line of the arrangement (i.e. a line not from N) intersects the lines from N in at least m points, and there are at most two lines which intersect the lines from N in exactly m points. In the case (i) the inequality is held because

$$f \geqslant m^2 + 2m - 1 + (m+1)(n-2m) - 2 \geqslant (m+1)(n-m).$$

Now consider the case (ii). If the arrangement contains at most one line which intersects the lines from N in m points, then the number of regions can be bounded using the number of intersection points of the other lines as follows:

$$f \geqslant m^2 + (m+1)(n-2m+1) - 1 = (m+1)(n-m).$$

If the arrangement contains two lines intersecting N in m points each, and their intersection point does not belong to the lines from N, then we similarly obtain $f \geqslant (m+1)(n-m)$.

The remaining case is when the arrangement contains two lines intersecting N in m points each, and their intersection point belongs to a line of N. Denote these lines by l_{2m} and l_{2m+1}, and their intersection point by R. Let us prove in this case that every other line of the arrangement intersects $N \cup l_{2m} \cup l_{2m+1}$ in at least $m+2$ points. It will follow from this fact that

$$f \geqslant m^2 + 2m + (m+2)(n-2m-1) \geqslant (m+1)(n-m)$$

which will end the proof in the case (ii).

Let us denote the lines passing through the point P in the sequence order, counting from the line PQ, by $l_1, \dots, l_{m-1}$. Similarly denote the lines passing through Q in the sequence order, counting from PQ, by $l_m, \dots, l_{2m-2}$. Denote by $A_{i,j}$ the point of intersection of lines l_i and l_{m-1+j}, where $1 \leqslant i, j \leqslant m-1$. Without loss of generality, we may assume that the line PQ is infinite, that the line l_{2m} passes through the points $A_{i,i}$ for $1 \leqslant i \leqslant m-1$, and that the line l_{2m+1} passes through the points $A_{i,m-i}$ for $1 \leqslant i \leqslant m-1$. Then the lines l_{2m} and l_{2m+1} intersect in the point $A_{\frac{m}{2},\frac{m}{2}}$, and m is even. Consider an arbitrary remaining line l of the arrangement. It intersects N in at least $m+1$ points. If at least one point of intersection of l with l_{2m} or with l_{2m+1} does not belong to the lines from N, then l intersects $N \cup l_{2m} \cup l_{2m+1}$ in at least $m+2$ points.

Let us note that if the line l passes through the point $A_{\frac{m}{2},\frac{m}{2}}$, then the intersection points of l with the lines $l_{\frac{m}{2}-1}$ and $l_{\frac{m}{2}+1}$ or with the lines $l_{m-1+\frac{m}{2}-1}$ and $l_{m-1+\frac{m}{2}+1}$ differ from the points A_{ij}. Hence, the line l intersects N in at least $m+2$ points.

Without loss of generality, we may assume that the intersection point of lines l and PQ and the intersection point of the lines l_{2m} and PQ belong to one line segment PQ of the line PQ (in the projective line, two points divide the line into two segments). Then the line l intersects l_{2m+1} in a point $A_{a,b}$ with $|a-b| \geqslant 2$. Hence, the line l passes through at most $\min\{a,b\} + \min\{m-1-a, m-1-b\} \leqslant m-3$ points of form $A_{i,j}$. Consequently, the line l intersects N in at least $m+2$ points.

The statement is not true for $n = 2m+1$. The counterexample for every even $m \geqslant 4$ can be constructed as follows: m lines pass through each of P and Q, the line PQ is common. Two more lines intersect these $2m-1$ lines in m points each. It gives $f = m^2+2m < (m+1)^2$.

Problem 23*. If we add the infinite line to the arrangement, the number of regions does not change, the number of lines n increases by 1, and we obtain $p+1$ concurrent lines instead of p parallel lines. It gives us the following statement:

The number f can be the number of regions of the projective plane divided by n lines if and only if f belongs to at least one of the intervals $[a(n,m),\ b(n,m)]$, $n \geqslant m \geqslant 2$, where

$$b(n,m) = m(n-m+1) + C^2_{n-m}\ , \quad a(n,m) = b(n,m) - \min\{m-1, C^2_{n-m}\}.$$

As before, a *gap* is an interval $(b(n,m), a(n,m-1))$ which contains at least one integer. It follows from problem 8 that $L(n) = \left[\sqrt{2n-5\frac{3}{4}} - \frac{1}{2}\right]$. In the sequel we denote $L(n)$ just by L, omitting n. Taking into consideration problem 13, it remains to prove that the number of regions cannot belong to the two last gaps numbered $L-1$ and L.

Suppose that there exists an arrangement such that the value f belongs to the $(L-1)$th gap, namely to the interval

$$\left((n-L+2)(L-1) + C^2_{L-2},\ \ L(n-L+1)\right).$$

It follows from problem 9 that $m = \max\{p+1,\ q\} \leqslant L-1$. Apply problem 21 for $M = L-1$:

$$f \geqslant \frac{2(n^2-n)}{L+2} \geqslant L(n-L+1).$$

In the last inequality, we used that $n \geqslant \frac{L^2+L}{2} + 3$ as follows:

$$n^2 - n \geqslant (n + \frac{L}{2} - 3)(n-L+1) \geqslant \left(\frac{L^2+2L}{2}\right)(n-L+1).$$

It contradicts with the fact that f belongs to the $(L-1)$th gap.

Now let us assume that there exists an arrangement of lines in the projective plane such that the number of regions belongs to the Lth gap:

$$f \in \left(L(n-L+1) + C^2_{L-1},\ (L+1)(n-L)\right).$$

By problem 4 we may assume that $L \geqslant 4$. Using problem 9, we have $m \leqslant L$. Consider 3 cases.

(1) If $m < L$, then by problem 21(d) for $M = L-1$ we obtain

$$f > \frac{2(n^2-n)}{L+2} \geqslant (L+1)(n-L),$$

because it follows from $n \geqslant \frac{n^2+n}{2} + 3$ that

$$\frac{(L+2)(L+1)}{2}(n-L) \leqslant (n+L-2)(n-L) \leqslant n^2 - 2n.$$

(2) If $m = L$ and $t_m = 1$, then throw out one line passing through a point which belonged to m lines. This operation decreases the number of regions. We can apply the inequality from problem 21(d) for $M = L - 1$ to the remaining arrangement of $n - 1$ lines and obtain

$$f \geqslant 2\left(\frac{n^2 - 3n + 2L}{L + 2}\right) \geqslant (L + 1)(n - L).$$

Let us prove the last inequality. If $n \geqslant \frac{L^2+L}{2} + 4$, then

$$n^2 - 3n + 2L \geqslant (n - L)(n + L - 3) \geqslant (n - L)\left(\frac{L^2 + 3L + 2}{2}\right).$$

The case $n = \frac{L^2+L}{2} + 3$ can be checked directly.

(3) If $m = L$ and $t_m \geqslant 2$, then $n \geqslant 2m + 2$, and it follows from the preceding case that $f \geqslant (L + 1)(n - L)$.

Problem 24. (a) Answer: it is the set of values from the main theorem for an arrangement of n lines in the plane. Let us consider a circle centered in the common intersection point and make an inversion with respect to this circle. We obtain an arrangement of lines, and the number of regions did not change. Conversely, if an arrangement of lines is given, then we can make an inversion with respect to some circle such that its circle does not belong to any line. We will get an arrangement of circles with the same number of regions.

(b) Answer: it is the set of doubled values from the main theorem for the projective plane. Consider the projection of the sphere with center O to the plane α tangent to the sphere in its south pole. Then a point X of the sphere is mapped to the point of intersection of the line OX with the plane α. If OX is parallel to α then X is mapped to the infinite point of direction OX. Under this projection, the big circles of the sphere are mapped to the lines of the projective plane, and the pair of opposite (symmetric wrt the center) regions of the sphere are mapped to the same region of the projective plane. Hence, the number of regions of the sphere equals the doubled number of regions of the projective plane divided by the corresponding arrangement of lines. Conversely, for every given arrangement of lines on the projective plane, we can construct an arrangement of big circles on the sphere via the inverse projection from the projective plane to the sphere, and this operation will double the number of regions.

Problem 25. (a) Consider the pair “line AB, point C not on this line”, such that the points A, B, C belong to the given set and the distance from the point C to the line AB is minimal among all such pairs. Suppose that the line AB contains a point D from the given set. Without loss of generality, we may suppose that D is between A and B and that the angle $ADC \geqslant 90°$. Then the distance between D and AC is less than the distance between C and AB, which contradicts the choice of points A, B, and C. Hence, the line AB does not contain the initial points except A and B.

(b) *Solution 1.* Suppose the contrary, i.e. each intersection point belongs to at least three given lines. It can be easily seen that we are not in the case when there are only two directions of lines. Now we can assume that the arrangement contains three pairwise non-parallel lines which are not concurrent. For all such triples of lines, consider the triangle formed by these three lines, and choose among them the triangle ABC of the minimal nonzero area. Since at least three points pass through each of A, B, and C, and using the fact that the area of ABC is minimal, we obtain that the lines passing through A, B, and C parallel to BC, CA, and AB, respectively, belong to the arrangement. Let us denote by A_1, B_1, and C_1

the intersection points of these three lines. The vertices of ABC are the midpoints of sides of $A_1B_1C_1$. Hence, the areas of triangles A_1BC, B_1CA, and C_1AB also equal the area of ABC. If we similarly continue with the triangles A_1BC, B_1CA, and C_1AB, we obtain the infinite number of initial lines, contradiction.

Solution 2. Let us reduce the problem to the case (a) via the polar duality. Consider a circle whose center does not belong to the lines of the arrangement. Make a polar transformation with respect to this circle. For every point, it puts into correspondence a line, and for every line, it puts into correspondence a point. The arrangement of n lines is mapped to the arrangement of n non-collinear points. Using (a), there exists a line l containing exactly two poles A and B. If we make this polar transformation once more (i.e. the inverse), we obtain that there exist exactly two lines of the arrangement passing through the pole L of the line l, namely the polar duals a and b of points A and B. Here we used the fact that the polar dual of the intersection point of two lines is the line which passes through the poles of these lines.

Problem 26. Consider a graph on the projective plane: its vertices are the intersection points of lines, the edges are the line segments which do not contain other intersection points in their interior. Let us denote by v and e the numbers of vertices and edges of this graph, respectively. We denot by f, as usually, the number of regions of the projective plane. Without loss of generality let us assume that one of lines is infinite (if not, find a central projection mapping one of the lines to the infinite line, this operation does not change the values v, e, f, t_i, and p_j). Note that if a connected graph on the projective plane contains all the infinite points, then the Euler formula can be written as follows: $v - e + f = 1$. Since $t_n = 0$, all the regions of the projective plane formed by the graph are bounded by at least three edges. Hence, $p_2 = 0$. Note that then

$$v = \sum_{i \geqslant 2} t_i, \quad e = \sum_{i \geqslant 2} it_i = \frac{1}{2} \sum_{j \geqslant 3} jp_j, \quad f = \sum_{j \geqslant 3} p_j.$$

Consequently,

$$3 = 3f - (2e+e) + 3v = 3\sum_{j \geqslant 3} p_j - \left(\sum_{j \geqslant 3} jp_j + \sum_{i \geqslant 2} it_i \right) + 3\sum_{i \geqslant 2} t_i = \sum_{j \geqslant 3} (3-j)p_j + \sum_{i \geqslant 2} (3-i)t_i.$$

Problem 27. Answer: $\dfrac{n: \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7 \;\; 8 \;\; 9}{t_2: \;\; 3 \;\; 3 \;\; 4 \;\; 3 \;\; 3 \;\; 4 \;\; 6}$. The example for $n \geqslant 6$ is constructed in problem 28.

If follows from the problem 26 that $t_2 \geqslant 3$.

(i) Show that for 5 lines we have $t_2 \geqslant 4$. It follows from problem 19 that $10 = t_2 + 3t_3 + 6t_4$. Hence, t_2 is not divisible by 3.

(ii) Show that for 8 lines we have $t_2 \geqslant 4$. If $t_5 + t_6 + t_7 = 1$, then $t_2 \geqslant 4$. Otherwise it follows from problem 19 that $28 = t_2 + 3t_3 + 6t_4$. Hence, t_2 is not divisible by 3.

(iii) Show that for 9 lines we have $t_2 \geqslant 6$. If $t_6 + t_7 + t_8 = 1$, then it follows from problem 26 that $t_2 \geqslant 6$. If $t_5 = 1$, then five concurrent lines intersect four other lines in at least $5 \cdot 4 - 2C_4^2 = 8$ points of multiplicity 2, i.e. $t_2 \geqslant 8$. If $t_5 + t_6 + t_7 + t_8 = 0$, then

$$36 = t_2 + 3t_3 + 6t_4 \quad \text{and} \quad t_2 \geqslant 3 + t_4.$$

Now using $t_2 < 6$ from problem 26 we obtain that $t_2 = 3$, $t_3 = 11$, $t_4 = 0$, and all the regions are triangular, $p_3 = 26 = f$. It remains to show that such a configuration does not exist.

Problem 28. (a) Take all the sides of a regular $\frac{n}{2}$gon and its $\frac{n}{2}$ symmetry axes. For these n lines $t_2 = \frac{n}{2}$.

(b) If $n = 4k + 1$, then add the infinite line to the example for $4k$ lines. If $n = 4k + 3$, then take the preceding example for $4k + 4$ lines and remove one line which does not pass through the vertices of the $2k + 2$-gon. In both cases we obtain n lines and $t_2 = 3k$.

Problem 29$^\bigstar$. The proof can be found in the paper "On the number of ordinary lines determined by n points", after L. M. Kelly and W. O. J. Moser in the "Canadian Journal of Mathematics", 1958, pp. 210-219.

Problem 30$^\bigstar$. The best known result is: $t_2 \geqslant \frac{6}{13}n$ for $n \geqslant 8$. The proof can be found in the paper "There exist $\frac{6n}{13}$ ordinary points", after J. Csima and E. T. Sawyer in the journal "Discrete and Computational Geometry", 1993, **9** pp. 187-202.

Problem 31*. If there exist two poinst such that every line passes through at least one of these points, then the required inequality can be shown as follows. Suppose that a lines pass through the first of these points and that b pass through the second. If $a + b = n$, then

$$a \geqslant 3,\ b \geqslant 3, \quad t_2 = ab, \quad \sum_{i\geqslant 4}\left(2i - 7\frac{1}{2}\right) t_i \leqslant 2a + 2b - 15.$$

If $a + b = n + 1$, then

$$a \geqslant 4,\ b \geqslant 4, \quad t_2 = (a-1)(b-1), \quad \sum_{i\geqslant 4}\left(2i - 7\frac{1}{2}\right) t_i = 2a + 2b - 15.$$

Now let us assume that there are no two points such that every line passes through at least one of them. Apply all the three items of problem 32:

$$3p_4+\sum_{j\geqslant 5} jp_j \geqslant z+x = z+\left(y + 2t_2 - \sum_{i\geqslant 3} it_i\right) \geqslant \frac{3}{2}\sum_{i\geqslant 3} t_i+2t_2-\sum_{i\geqslant 3} it_i = 2t_2-\sum_{i\geqslant 3}\left(i - \frac{3}{2}\right) t_i.$$

Using problem 26, we obtain:

$$\sum_{i\geqslant 2}(9-3i)t_i = 9 + 3p_4 + \sum_{j\geqslant 5}(3j-9)p_j.$$

Note that $3j - 9 \geqslant j$ for $j \geqslant 5$ and $p_j \geqslant 0$. Hence, we have

$$\sum_{i\geqslant 2}(9-3i)t_i = 9 + 3p_4 + \sum_{j\geqslant 5}(3j-9)p_j \geqslant 3p_4 + \sum_{j\geqslant 5} jp_j \geqslant 2t_2 - \sum_{i\geqslant 3}\left(i - \frac{3}{2}\right) t_i,$$

which implies the required inequality for t_is.

Problem 32. Consider the corresponding graph: its vertices are the intersection points (colored in one of two colors), the edges are the line segments which do not contain other intersection points except their ends.

(a) Every red vertex is a red edge of four edges. Every red edge has two red edges, every non-colored edge has one red edge, blue edges do not have red ends. There are t_2 red vertices. Hence,

$$4t_2 = 2x + \sum_{i\geqslant 2} it_i - x - y,$$

which gives the required equality.

(b) Consider an arbitrary blue point O and delete all the lines of the arrangement passing through it. The remaining lines are not concurrent, hence they divide the projective plane into regions, each of them bounded by at least three edges. The point O belongs to one of such regions. By a case-by-case consideration, we see that at least one of the three possibilities holds for O:

(1) In the initial graph, the point O is incident to at least three blue edges.

(2) The point O is incident to at least two blue edges, and O is the vertex of the boundary of at least one green region.

(3) The point O is the vertex of the boundary of at least two green regions.

It means that for every blue vertex, the sum of the number of blue edges incident to it with the doubled number of green regions incident to it is at least 3. If we sum up these sums for all the blue vertices, we obtain $2y + 2s \geqslant \sum_{i\geqslant 3} t_i$.

(c) For a green region u, let us define by $x(u)$ and $s(u)$ the number of red edges and the number of blue vertices at the boundary of u, respectively. For a green region u let

$$d(u) = \begin{cases} 0, & \text{if} \quad s(u) \geqslant 1; \\ 1, & \text{if} \quad s(u) = 0. \end{cases}$$

It is easy to prove that if a green region u is bounded by j edges, then

$$s(u) \leqslant (j-1) - x(u) + d(u).$$

Denote by X and by D the sums of $x(u)$ and $d(u)$, respectively, by all the green regions. Summing up the inequality obtained above for all the green regions, we get

$$s \leqslant \sum_{j\geqslant 4} (j-1)p_j \; - X + D.$$

Note that every red edge is incident to at least one green region (since $t_n = t_{n-1} = 0$). A red edge is called dark red if it is adjacent to two green regions. Let us denote by x_1 the number of dark-red edges. Then $X = x + x_1$. Let us distinguish the 4-hedral regions such that all their vertices are red (all the distinguished regions are green). It is easy to show that every distinguished region is incident to at least two dark-red enges. Hence, the number of distinguished regions is at most x_1. It follows that $D \leqslant x_1 + \sum_{j\geqslant 5} p_j$. Now merge all the inequalities.