Shapiro's inequality

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1 *Shapiro's inequality*

In October, 1954 the American Mathematical Monthly published the following problem of Harold Shapiro

Prove the following inequality for positive numbers $x_1,\ x_2,\ \ldots,\ x_n$:

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\ldots+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2}\geqslant\frac{n}{2}\,, \tag{1}$$

the equality holds only if all the denominators are equal.

In contrast to, say, "Kvant" magazine, it was allowed to publish problems in the Monthly, which were not solved by the proposer, and the readers had not been informed about this nuance. This time the situation was exactly like that. The author had a solution for partial cases $n=3$ and 4 only.

In the following problems we can replace the condition that all the x_k's are positive with the condition that all the x_k's are nonnegative and all the denominators are nonzero. Indeed, if the inequality is proven for positive numbers, then it is not difficult to deduce the inequality for nonnegative numbers (and nonzero denominators). Let

$$f(x_1,x_2,\ldots,x_n)=\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\ldots+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2}\,.$$

1.1. Prove the inequality (1) for $n=3,\ 4,\ 5,\ 6$.

1.2. Prove that the inequality (1) is wrong
a) for $n=20$; b) for $n=14$; c) for $n=25$.

1.3. Prove the inequality (1) for monotonic sequences.

1.4. Prove that if the inequality (1) does not hold for $n=m$, then it does not hold for $n=m+2$.

1.5. Prove that if the inequality (1) does not hold for $n=m$, where m is odd, then it does not hold for all $n>m$.

1.6. Prove the inequality (1) for $n=8,\ 10,\ 12$ and for $n=7,\ 9,\ 11,\ 13,\ 15,\ 17,\ 19,\ 21,\ 23$. Due to the statement of the previous problem it is sufficient to prove the inequality only for $n=12$ and $n=23$.

1.7. Prove that $f(x_1,x_2,\ldots,x_n)+f(x_n,x_{n-1},\ldots,x_1)\geqslant n\,.$

1.8. Assume that the function $f(x_1,x_2,\ldots,x_n)$ has a local minimum in the point $(a_1,a_2,\ldots,a_n)$, $a_1,\ a_2,\ \ldots,\ a_n>0$.
a) Prove that $f(a_1,a_2,\ldots,a_n)=n/2$ if n is even.
b*) Prove the same statement for odd n.
c) Use the statements пїSa) and b) to prove the inequality for $n=7$ and $n=8$.

1.9. Prove the inequality $f(x_1,x_2,\ldots,x_n)\geqslant cn$ for the following values of the constant c:
a) $c=1/4$; b) $c=\left(\sqrt{2}-1\right)$; c) $c=5/12$.

2 *Useful and related inequalities*

Prove the following inequalities assuming that all the x_k's are positive. Prove that the constants printed in bold can not be decreased (for each n).

2.1. Mordell's inequality.
a) $\displaystyle\left(\sum_{k=1}^{n}x_k\right)^2\geqslant\mathbf{min}\left\{\frac{\mathbf{n}}{\mathbf{2}},\mathbf{3}\right\}\cdot\sum_{k=1}^{n}x_k(x_{k+1}+x_{k+2})$.
b) Find all n-tuples $x_1,\ x_2,\ \ldots,\ x_n$ such that the equality is achieved.

2.2. $\displaystyle\left(\sum_{k=1}^{n}x_k\right)^2\geqslant\mathbf{min}\left\{\frac{\mathbf{n}}{\mathbf{3}},\frac{\mathbf{8}}{\mathbf{3}}\right\}\cdot\sum_{k=1}^{n}x_k(x_{k+1}+x_{k+2}+x_{k+3})$.

2.3. пїSa) Prove that for $n\leqslant 8$

$$\frac{x_1}{x_2+x_3+x_4}+\frac{x_2}{x_3+x_4+x_5}+\ldots+\frac{x_{n-1}}{x_n+x_1+x_2}+\frac{x_n}{x_1+x_2+x_3}\geqslant\frac{n}{3}\,.$$

b) For which $n>8$ this inequality is also true?

2.4. $(x_1+x_2+\ldots+x_n)^2 \geqslant 4(x_1x_2+x_2x_3+\ldots+x_{n-1}x_n+x_nx_1)\,; \quad n\geqslant 4.$

2.5. $\displaystyle\sum_{k=1}^{n}\frac{x_k}{x_{k+1}+x_{k+2}} \geqslant \sum_{k=1}^{n}\frac{x_{k+1}}{x_k+x_{k+1}}.$

2.6. $\dfrac{x_1}{x_n+x_2}+\dfrac{x_2}{x_1+x_3}+\ldots+\dfrac{x_{n-1}}{x_{n-2}+x_n}+\dfrac{x_n}{x_{n-1}+x_1}\geqslant \mathbf{2}\,; \quad n\geqslant 4.$

2.7. $\dfrac{x_1+x_2}{x_1+x_3}+\dfrac{x_2+x_3}{x_2+x_4}+\ldots+\dfrac{x_{n-1}+x_n}{x_{n-1}+x_1}+\dfrac{x_n+x_1}{x_n+x_2}\geqslant \mathbf{4}\,; \quad n\geqslant 4.$

2.8. $\dfrac{x_1}{x_n+x_3}+\dfrac{x_2}{x_1+x_4}+\ldots+\dfrac{x_{n-1}}{x_{n-2}+x_1}+\dfrac{x_n}{x_{n-1}+x_2}\geqslant \mathbf{3}\,; \quad n\geqslant 4.$

2.9. $\dfrac{x_2+x_3}{x_1+x_4}+\dfrac{x_3+x_4}{x_2+x_5}+\ldots+\dfrac{x_n+x_1}{x_{n-1}+x_2}+\dfrac{x_1+x_2}{x_n+x_3}\geqslant \mathbf{6}\,; \quad n\geqslant 6.$

2.10. $\dfrac{x_1+x_2}{x_1+x_4}+\dfrac{x_2+x_3}{x_2+x_5}+\ldots+\dfrac{x_{2004}+x_1}{x_{2004}+x_3}\geqslant \mathbf{6}\,.$

2.11. $\dfrac{x_1}{x_n+x_4}+\dfrac{x_2}{x_1+x_5}+\ldots+\dfrac{x_{n-1}}{x_{n-2}+x_2}+\dfrac{x_n}{x_{n-1}+x_3}\geqslant \mathbf{4}$, where $n>5$ is even.

2.12. $\displaystyle\sum_{k=1}^{n}\frac{x_k^2}{x_{k+1}^2-x_{k+1}x_{k+2}+x_{k+2}^2}\geqslant\left[\frac{\mathbf{n+1}}{\mathbf{2}}\right].$

3 *After the intermediate finish*

1.10. a) Prove that for each n there exists $q_n>1$, such that for all real $x_1,\,x_2,\,\ldots,\,x_n\in[\frac{1}{q_n};q_n]$ the inequality (1) holds.

b*) Is it possible to choose $q>1$, such that for all integers $n>0$ and for all $x_i\in[\frac{1}{q};q]$ the inequality (1) holds?

1.11. Let $S=f(x_1,x_2,\ldots,x_n)$ be the left hand side of Shapiro's inequality. Denote by $a_1,\,a_2,\,\ldots,\,a_n$ the numbers $x_2/x_1,\,x_3/x_2,\,\ldots,\,x_n/x_{n-1},\,x_1/x_n$, arranged in increasing order.

a) Prove that $S\geqslant\frac{1}{a_1(1+a_n)}+\frac{1}{a_2(1+a_{n-1})}+\ldots+\frac{1}{a_n(1+a_1)}$;

b) Let $b_k=\begin{cases}\dfrac{1}{a_ka_{n+1-k}}, & a_ka_{n+1-k}\geqslant 1\\[2mm] \dfrac{2}{a_ka_{n+1-k}+\sqrt{a_ka_{n+1-k}}}, & a_ka_{n+1-k}<1.\end{cases}$ Prove that $2S\geqslant b_1+b_2+\ldots+b_n$;

c) Let g be the maximal convex function that does not exceed both functions e^{-x} пїS $2(e^x+e^{x/2})^{-1}$. Prove that $2S\geqslant g(\ln(a_1a_n))+g(\ln(a_2a_{n-1}))+\ldots+g(\ln(a_na_1))\geqslant ng(0)$.

d) Prove that for each $\lambda>g(0)$ there exist a nonnegative integer n and positive numbers $x_1,\,x_2,\,\ldots,\,x_n$, such that $S\leqslant\lambda n$.

Solutions

1.1. $\underline{n=3}$. Let $S = x_1 + x_2 + x_3$. It is easy to see that the function $f(t) = \frac{t}{S-t}$ is convex on the interval $[0; S)$. Apply the Jensen inequality to it:

$$\frac{f(x_1) + f(x_2) + f(x_3)}{3} \geqslant f\left(\frac{x_1 + x_2 + x_3}{3}\right) = f\left(\frac{S}{3}\right) = \frac{1}{2}.$$

We are done.

$\underline{n=4}$. This inequality is cyclic. Write down the values of x_i's successively at the vertices of a square. Then on each diagonal put an arrow leading from the smaller value to the greater one. Notice that there is a side of the square with two tails on it. Re-number the x_i's in such a manner that this side becomes x_4x_1. Now we may assume that $x_1 \geqslant x_3$, $x_4 \geqslant x_2$. For the variables with these restrictions the following inequality is true:

$$\frac{x_1}{x_2 + x_3} + \frac{x_3}{x_4 + x_1} \geqslant \frac{x_1}{x_4 + x_3} + \frac{x_3}{x_2 + x_1}\,.$$

Indeed, re-write it in the following way:

$$x_1\Big(\frac{1}{x_2 + x_3} - \frac{1}{x_4 + x_3}\Big) \geqslant x_3\Big(\frac{1}{x_2 + x_1} - \frac{1}{x_4 + x_1}\Big)\,.$$

Reduce both hands to a common denominator, cancel $x_4 - x_2$ in both hands (if $x_4 - x_2 = 0$, we already have the equality), and multiply both hands to the product of denominators. We obtain the evident (since $x_1 \geqslant x_3$) inequality

$$x_1(x_2 + x_1)(x_4 + x_1) \geqslant x_3(x_2 + x_3)(x_4 + x_3)\,.$$

Use it to prove Shapiro's inequality:

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\frac{x_3}{x_4+x_1}+\frac{x_4}{x_1+x_2} \geqslant \frac{x_1}{x_4+x_3}+\frac{x_2}{x_3+x_4}+\frac{x_3}{x_2+x_1}+\frac{x_4}{x_1+x_2} = \frac{x_1+x_2}{x_3+x_4}+\frac{x_3+x_4}{x_1+x_2} = a+a^{-1} \geqslant 2\,.$$

$\underline{n=5}$. Notice that the function $f(t) = 1/(S-t)$ is convex on the interval $[0; S)$. So we can apply the Jensen inequality with $n = 5$:

$$a_1 f(t_1) + a_2 f(t_2) + a_3 f(t_3) + a_4 f(t_4) + a_5 f(t_5) \geqslant f(a_1t_1 + a_2t_2 + a_3t_3 + a_4t_4 + a_5t_5)\,, \tag{2}$$

where $a_i \geqslant 0$, $\sum a_i = 1$. Take $a_i = \frac{x_i}{S}$, and let $t_i = x_i + x_{i-1} + x_{i-2}$, $i = 1, \ldots, 5$ (we assume that the variables are enumerated cyclically: $x_0 = x_5$, $x_{-1} = x_4$). Then $f(t_i) = \frac{1}{S-t_i} = \frac{1}{x_{i+1}+x_{i+2}}$, and it means that the left-hand side of inequality (2) coincides with the left-hand side of Shapiro's inequality. Now consider the right-hand side of 2:

$$\frac{1}{S - \sum_{i=1}^{5} a_it_i} = \frac{1}{S - \sum_{i=1}^{5} \frac{x_i}{S}(x_i + x_{i-1} + x_{i-2})} = \frac{S}{S^2 - \sum_{i=1}^{5} x_i(x_i + x_{i-1} + x_{i-2})}.$$

Open the brackets. It is easy to see that the denominator is the sum of pairwise products of the set of variables x_i. Since the initial inequality is homogeneous, we may assume that $S = x_1 + x_2 + x_3 + x_4 + x_5 = 1$. Now the right-hand side of inequality (2) is the inverse number to the sum of pairwise products of the variables x_i, satisfying one condition $x_1 + x_2 + x_3 + x_4 + x_5 = 1$. The right-hand side reaches its minimum when the sum of pairwise products reaches its maximum. It is well-known that for it all the variables should be equal. But the right-hand side equals $5/2$ in this point.

The analogous proof also works for $n = 4$.

$\underline{n=6}$. Proceed as above. The function $f(t) = 1/(S-t)$ is convex on the interval $[0; S)$. So we can apply the Jensen inequality with $n = 6$:

$$\sum_{i=1}^{6} a_i f(t_i) \geqslant f\Big(\sum_{i=1}^{6} a_it_i\Big).$$

Let $a_i = \frac{x_i}{S}$, $t_i = x_i + x_{i-1} + x_{i-2} + x_{i-3}$, $i = 1, \ldots, 6$ (we assume that the variables are enumerated cyclically: $x_0 = x_6$, $x_{-1} = x_5$, $x_{-2} = x_4$). Then $f(t_i) = \frac{1}{S-t_i} = \frac{1}{x_{i+1}+x_{i+2}}$, and this means that the left-hand side of the inequality (1.1) coincides with the left-hand side of Shapiro's inequality. Now consider the right-hand side of (1.1):

$$\frac{1}{S - \sum_{i=1}^{6} a_it_i} = \frac{1}{S - \sum_{i=1}^{6} \frac{x_i}{S}(x_i + x_{i-1} + x_{i-2} + x_{i-3})} = \frac{S}{S^2 - \sum_{i=1}^{6} x_i(x_i + x_{i-1} + x_{i-2} + x_{i-3})}.$$

Open the brackets. It is easy to see that the denominator is the sum of pairwise products of the variables x_i's but the products x_1x_4, x_2x_5, and x_3x_6. This sum can be re-written as $(x_1+x_4)(x_2+x_5)+(x_1+x_4)(x_3+x_6)+(x_2+x_5)(x_3+x_6)$. Denote $A = x_1 + x_4$, $B = x_2 + x_5$, $C = x_3 + x_6$. The right-hand side of (1.1) can be re-written as

$$\frac{A+B+C}{AB+BC+AC}\,. \tag{3}$$

Since the initial inequality is homogeneous, we may assume that $S = x_1 + x_2 + x_3 + x_4 + x_5 = A + B + C = 1$. Now it is clear that the expression (3) is greater than or equal to 3, since $(A + B + C)^2 \geqslant 3(AB + BC + AC)$. пїЅпїЅпїЅ

Remark. Unfortunately, this method does not work for $n > 6$.

Second solution. Apply the Cauchy-Bunyakovsky inequality to the sets of numbers

$$\sqrt{\frac{x_1}{x_2+x_3}},\quad \sqrt{\frac{x_2}{x_3+x_4}},\quad \ldots,\quad \sqrt{\frac{x_n}{x_1+x_2}} \qquad \text{and}$$
$$\sqrt{x_1(x_2+x_3)},\quad \sqrt{x_2(x_3+x_4)},\quad \ldots,\quad \sqrt{x_n(x_1+x_2)}\quad .$$

We obtain

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\ldots+\frac{x_n}{x_1+x_2} \geqslant \frac{(x_1+x_2+\ldots+x_n)^2}{x_1(x_2+x_3)+x_2(x_3+x_4)+\ldots+x_n(x_1+x_2)}\,.$$

Use Mordell's inequality (problem 2.1). When $n \leqslant 6$, it gives us that the right-hand side of this inequality is greater than or equal to $n/2$.

1.2. a) [22] Take as x_1, x_2, …, x_{20} numbers

$$\begin{array}{cccccccccc} 1+5\varepsilon, & 6\varepsilon, & 1+4\varepsilon, & 5\varepsilon, & 1+3\varepsilon, & 4\varepsilon, & 1+2\varepsilon, & 3\varepsilon, & 1+\varepsilon, & 2\varepsilon, \\ 1+2\varepsilon, & \varepsilon, & 1+3\varepsilon, & 2\varepsilon, & 1+4\varepsilon, & 3\varepsilon, & 1+5\varepsilon, & 4\varepsilon, & 1+6\varepsilon, & 5\varepsilon. \end{array}$$

Then $f(x_1, \ldots, x_{20}) < 10 - \varepsilon^2 + c\varepsilon^3 < 10$ for some c and small enough ε.

b) [27] Take as x_1, x_2, …, x_{14} numbers

$$1+7\varepsilon,\ 7\varepsilon,\ 1+4\varepsilon,\ 6\varepsilon,\ 1+\varepsilon,\ 5\varepsilon,\ 1,\ 2\varepsilon,\ 1+\varepsilon,\ 0,\ 1+4\varepsilon,\ \varepsilon,\ 1+6\varepsilon,\ 4\varepsilon.$$

Then $f(x_1, \ldots, x_{20}) < 7 - 2\varepsilon^2 + c\varepsilon^3 < 7$ for some c and small enough ε.

An alternative example [24]:

$$0,\ 42,\ 2,\ 42,\ 4,\ 41,\ 5,\ 39,\ 4,\ 38,\ 2,\ 38,\ 0,\ 40\,.$$

c) [10], [18]. Take

$$0,\ 85,\ 0,\ 101,\ 0,\ 120,\ 14,\ 129,\ 41,\ 116,\ 59,\ 93,\ 64, 71,\ 63,\ 52,\ 60,\ 36,\ 58,\ 23,\ 58,\ 12,\ 62,\ 3,\ 71.$$

Alternatively, in [3] the following example is given:

$$32,\ 0,\ 37,\ 0,\ 43,\ 0,\ 50,\ 0,\ 59,\ 8,\ 62,\ 21,\ 55, 29,\ 44,\ 32,\ 33,\ 31,\ 24,\ 30,\ 16,\ 29,\ 10,\ 29,\ 4.$$

1.3. The statement of the problem is published in [13]. We present here a short nice solution.

Let $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n > 0$. Observe that the product of n fractions $\frac{x_k+x_{k+1}}{x_{k+1}+x_{k+2}}$ is equal to 1. Then by Cauchy inequality we conclude that

$$\sum_{k=1}^{n}\frac{x_k+x_{k+1}}{x_{k+1}+x_{k+2}} \geqslant n = \sum_{k=1}^{n}\frac{x_{k+1}+x_{k+2}}{x_{k+1}+x_{k+2}}.$$

Hence

$$\sum_{k=1}^{n}\frac{x_k}{x_{k+1}+x_{k+2}} \geqslant \sum_{k=1}^{n}\frac{x_{k+2}}{x_{k+1}+x_{k+2}} = \sum_{k=1}^{n}\frac{x_{k+1}}{x_k+x_{k+1}}. \tag{4}$$

Now we will apply the *rearranging inequality*: Let $a_1 \geqslant \ldots \geqslant a_n$ and $b_1 \geqslant \ldots \geqslant b_n$ be two sets of numbers. Then for each permutation k_1, …, k_n of numbers 1, …, n the following inequality holds

$$a_1b_1 + a_2b_2 + \ldots a_nb_n \geqslant a_1b_{k_1} + a_2b_{k_2} + \ldots a_nb_{k_n} \geqslant a_1b_n + a_2b_{n-1} + \ldots a_nb_1.$$

Use the rearranging inequality twice

$$\begin{aligned}\sum_{k=1}^{n}\frac{x_k}{x_{k+1}+x_{k+2}}&=\sum_{k=1}^{n-2}\frac{x_k}{x_{k+1}+x_{k+2}}+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2}\underset{(*)}{\geqslant}\\&\geqslant\sum_{k=1}^{n-2}\frac{x_k}{x_{k+1}+x_{k+2}}+\frac{x_{n-1}}{x_1+x_2}+\frac{x_n}{x_n+x_1}\underset{(**)}{\geqslant}\\&\geqslant\sum_{k=1}^{n}\frac{x_k}{x_k+x_{k+1}}.\end{aligned}$$

The inequality (*) here is the rearranging inequality for two pairs of numbers: $x_{n-1}\geqslant x_n$ and $\frac{1}{x_n+x_1}\geqslant\frac{1}{x_1+x_2}$; and the inequality (**) is the rearranging inequality for the sets $x_1, x_2, \ldots, x_{n-1}$ and $\frac{1}{x_1+x_2}, \frac{1}{x_2+x_3}, \ldots, \frac{1}{x_{n-1}+x_n}$ that have opposite ordering.

Thus

$$2\sum_{k=1}^{n}\frac{x_k}{x_{k+1}+x_{k+2}}\geqslant\sum_{k=1}^{n}\frac{x_k}{x_k+x_{k+1}}+\sum_{k=1}^{n}\frac{x_{k+1}}{x_k+x_{k+1}}=n.$$

For the decreasing set x_i the solution is similar because we do not use the order of the variables when we apply the Cauchy inequality, and for the rearranging inequalities we need the fact that the sets x_i and $\frac{1}{x_i+x_{i+1}}$ have different orderings.

1.4. [3] It is easy to see that $f_{n+2}(x_1,x_2,\ldots,x_n,x_1,x_2)=f_n(x_1,x_2,\ldots,x_n)+1$. Therefore if $f_n(x_1,x_2,\ldots,x_n)<n/2$, then $f_{n+2}(x_1,x_2,\ldots,x_n,x_1,x_2)<(n+2)/2$.

1.5. [3] Assume that $f_m(x_1,x_2,\ldots,x_m)<\frac{m}{2}$. For each k let us calculate the difference

$$\begin{aligned}&f_{m+1}(x_1,\ldots,x_k,x_k,x_{k+1},\ldots,x_m)-f_m(x_1,x_2,\ldots,x_m)-\frac{1}{2}=\\&\qquad=\frac{x_{k-1}}{2x_k}+\frac{x_k}{x_k+x_{k+1}}-\frac{x_{k-1}}{x_k+x_{k+1}}-\frac{1}{2}=\frac{(x_k-x_{k-1})(x_k-x_{k+1})}{2x_k(x_k+x_{k+1})}.\end{aligned}$$

If $(x_k-x_{k-1})(x_k-x_{k+1})\leqslant 0$, then

$$f_{n+1}(x_1,x_2,\ldots,x_k,x_k,x_{k+1},\ldots,x_m)<\frac{m+1}{2}$$

and we are done. If n is odd, we can always choose k such that $(x_k-x_{k-1})(x_k-x_{k+1})\leqslant 0$ because otherwise the product of the (odd number of) inequalities $(x_k-x_{k-1})(x_{k+1}-x_k)<0$ for all k is

$$(x_2-x_1)^2(x_3-x_2)^2\ldots(x_m-x_{m-1})^2(x_1-x_m)^2<0\,.$$

Thus if for odd n the Shapiro inequality is wrong then for $n+1$ it is wrong, too. It remains to apply the statement of the previous problem.

1.6. [7, 8]

1.7. [28] Let $y_k=x_k+x_{k+1}$. Then

$$\frac{x_1+x_4}{x_2+x_3}+\frac{x_2+x_5}{x_3+x_4}+\ldots+\frac{x_n+x_3}{x_1+x_2}=\sum_{k=1}^{n}\frac{y_k-y_{k+1}+y_{k+2}}{y_{k+1}}=\sum_{k=1}^{n}\frac{y_k}{y_{k+1}}+\sum_{k=1}^{n}\frac{y_{k+2}}{y_{k+1}}-n\geqslant n,$$

because by Cauchy inequality each sum is at least n.

1.8. The statements a), b) were published in [21].

a) !!! This short proof is taken from [8].

Denote for brevity $a=(a_1,a_2,\ldots,a_n)$, $x=(x_1,x_2,\ldots,x_n)$, and $u=(-1,1,-1,1,\ldots,-1,1)$.

Observe that

$$\frac{\partial f}{\partial x_k}(x)=\frac{1}{x_{k+1}+x_{k+2}}-\frac{x_{k-2}}{(x_{k-1}+x_k)^2}-\frac{x_{k-1}}{(x_k+x_{k+1})^2}.$$

It is easy to see that we have an identity

$$f(x+tu)=f(x)+t\sum_{k=1}^{n}(-1)^k\frac{\partial f}{\partial x_k}(x).$$

Since a is the minimum point, we have

$$\frac{\partial f}{\partial x_k}(a)=0.$$

Therefore $f(a+tu) = f(a)$ if all the coordinates of the point $a+tu$ are positive. Hence $a+tu$ is the minimum point of the function f as well. Hence,

$$\frac{\partial f}{\partial x_k}(a+tu) = 0.$$

So

$$\frac{1}{a_{k+1}+a_{k+2}} - \frac{a_{k-2}}{(a_{k-1}+a_k)^2} - \frac{a_{k-1}}{(a_k+a_{k+1})^2} = 0$$

and

$$\frac{1}{a_{k+1}+a_{k+2}} - \frac{a_{k-2}+t(-1)^{k-2}}{(a_{k-1}+a_k)^2} - \frac{a_{k-1}+t(-1)^{k-1}}{(a_k+a_{k+1})^2} = 0.$$

Subtract the first equality from the second:

$$\frac{t}{(a_{k-1}+a_k)^2} - \frac{t}{(a_k+a_{k+1})^2} = 0.$$

Therefore,

$$a_{k-1}+a_k = a_k + a_{k+1}.$$

and hence

$$a_1 = a_3 = a_5 = \cdots = a_{n-1} \quad \text{and} \quad a_2 = a_4 = a_6 = \cdots = a_n.$$

Thus, $f(a) = n/2$.

b) This short proof is taken from [7]. Denote for brevity $a = (a_1, a_2, \ldots, a_n)$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, $z = (z_1, z_2, \ldots, z_n)$, where $y_k = x_k + x_{k+1}$ and $z_k = 1/y_{n+1-k}$.

Set

$$S(x) = \frac{x_1}{x_2+x_3} + \frac{x_2}{x_3+x_4} + \ldots + \frac{x_{n-1}}{x_n+x_1} + \frac{x_n}{x_1+x_2} = \sum_{k=0}^{n-1} \frac{x_k}{y_{k+1}}.$$

Observe that

$$\frac{\partial f}{\partial x_k}(x) = \frac{1}{x_{k+1}+x_{k+2}} - \frac{x_{k-2}}{(x_{k-1}+x_k)^2} - \frac{x_{k-1}}{(x_k+x_{k+1})^2}.$$

It is easy to check the following identities:

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d} + \frac{\dfrac{a}{b^2} + \dfrac{c}{d^2}}{\dfrac{1}{b} + \dfrac{1}{d}}.$$

Hence,

$$\frac{x_{k-2}}{x_{k-1}+x_k} + \frac{x_{k-1}}{x_k+x_{k+1}} = \frac{x_{k-2}+x_{k-1}}{(x_{k-1}+x_k)+(x_k+x_{k+1})} + \frac{\dfrac{x_{k-2}}{(x_{k-1}+x_k)^2} + \dfrac{x_{k-1}}{(x_k+x_{k+1})^2}}{\dfrac{1}{x_{k-1}+x_k} + \dfrac{1}{x_k+x_{k+1}}} =$$

$$= \frac{y_{k-2}}{y_{k-1}+y_k} + \frac{z_{n-k} - \dfrac{\partial f}{\partial x_k}(x)}{z_{n-k+1}+z_{n-k+2}}.$$

Therefore,

$$2S(x) = S(y) + S(z) - \sum_{k=1}^{n} \frac{\dfrac{\partial f}{\partial x_k}(x)}{z_{n-k+1}+z_{n-k+2}}.$$

If x is a minimum point then we have $2S(x) = S(y) + S(z)$. Hence $S(x) = S(y) = S(z)$.

Let $u := (x_1 + x_2 + \cdots + x_n)/n$. Consider the transformation $M : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$M(x) = \Big(\frac{x_1+x_2}{2}, \frac{x_2+x_3}{2}, \ldots, \frac{x_n+x_1}{2}\Big).$$

Let $M_k(x)$ be its k-th iteration. Observe that $S(x) = S(y) = S(M(x)) = \cdots = S(M_k(x))$. It is clear that $\lim\limits_{k\to\infty} M_k(x) = (u, u, \ldots, u)$. Then

$$S(x) = \lim_{k\to\infty} S(M_k(x)) = S((u, u, \ldots, u)) = \frac{n}{2}.$$

c) [16], [7, 8]

1.9. These solutions are taken from [3].

пїЅ) The problem was presented at the Third USSR mathematical olympiad, 1969. Probably it was originally published in [14].

Let x_{i_1} be the maximal number among $x_1, x_2, \ldots, x_n$; x_{i_2} be the maximum of the two next numbers after x_{i_1} (i.e. of x_{i_1+1} and x_{i_1+2}); x_{i_3} be the maximum of the two next numbers after x_{i_2}, and so on. We will continue this sequence till the step number k when the maximum of the two next after x_{i_k} numbers is x_{i_1}.

It is clear that $k \geqslant n/2$. We have

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\ldots+\frac{x_n}{x_1+x_2} \geqslant \frac{x_{i_1}}{2x_{i_2}}+\frac{x_{i_2}}{2x_{i_3}}+\ldots+\frac{x_{i_k}}{2x_{i_1}}.$$

The last expression is at least $k/2$ by the Cauchy inequality therefore it is at least $n/4$.

b) Rewrite each of the fractions $\frac{x_k}{x_{k+1}+x_{k+2}}$, $k=1, 2, \ldots, n$, in the form

$$\frac{x_k}{x_{k+1}+x_{k+2}} = \frac{x_k+\frac{1}{2}x_{k+1}}{x_{k+1}+x_{k+2}} + \frac{\frac{1}{2}x_{k+1}+x_{k+2}}{x_{k+1}+x_{k+2}} - 1\,.$$

We obtain $2n$ fractions. Combine them by pairs: the first and the last, the second and the third, the fourth and the fifth and so on. Now estimate the sum of each pair from below

$$\frac{\frac{1}{2}x_k+x_{k+1}}{x_k+x_{k+1}}+\frac{x_k+\frac{1}{2}x_{k+1}}{x_{k+1}+x_{k+2}} \geqslant 2\sqrt{\frac{(\frac{1}{2}x_k+x_{k+1})(x_k+\frac{1}{2}x_{k+1})}{(x_k+x_{k+1})(x_{k+1}+x_{k+2})}} =$$

$$= 2\sqrt{\left(\frac{1}{2}+\frac{x_kx_{k+1}}{4(x_k+x_{k+1})^2}\right)\frac{x_k+x_{k+1}}{x_{k+1}+x_{k+2}}} > \sqrt{2}\cdot\sqrt{\frac{x_k+x_{k+1}}{x_{k+1}+x_{k+2}}}.$$

Since the product of n numbers $\sqrt{\frac{x_1+x_2}{x_2+x_3}}, \sqrt{\frac{x_2+x_3}{x_3+x_4}}, \ldots, \sqrt{\frac{x_n+x_1}{x_1+x_2}}$ equals 1, then by the Cauchy inequality their sum is at least n. Therefore $f(x_1, \ldots, x_n) \geqslant \sqrt{2}n - n = (\sqrt{2}-1)n$.

c) As in the previous solution rewrite each of the fractions $\frac{x_k}{x_{k+1}+x_{k+2}}$, $k=1, 2, \ldots, n$, in the form

$$\frac{x_k}{x_{k+1}+x_{k+2}} = \frac{x_k+\beta x_{k+1}}{x_{k+1}+x_{k+2}} + \alpha\cdot\frac{\beta x_{k+1}+x_{k+2}}{x_{k+1}+x_{k+2}} - \alpha\,,$$

where α and β are parameters chosen to make the equality true. For such a choice of α and β we need $\beta+\alpha\beta=\alpha$, i.e. $\beta=\alpha/(\alpha+1)$. Then

$$\frac{x_k+\beta x_{k+1}}{x_{k+1}+x_{k+2}} + \alpha\cdot\frac{\beta x_k+x_{k+1}}{x_k+x_{k+1}} \geqslant 2\sqrt{\alpha\frac{(x_k+\beta x_{k+1})(\beta x_k+x_{k+1})}{(x_k+x_{k+1})(x_{k+1}+x_{k+2})}} =$$

$$= 2\sqrt{\alpha\frac{\beta(x_k+x_{k+1})^2+(\beta-1)^2x_kx_{k+1}}{(x_k+x_{k+1})(x_{k+1}+x_{k+2})}} > 2\sqrt{\alpha\beta\frac{x_k+x_{k+1}}{x_{k+1}+x_{k+2}}} = \frac{2\alpha}{\sqrt{\alpha+1}}\cdot\sqrt{\frac{x_k+x_{k+1}}{x_{k+1}+x_{k+2}}}\,.$$

Therefore

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\ldots+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2} \geqslant \frac{2\alpha}{\sqrt{\alpha+1}}\left(\sqrt{\frac{x_1+x_2}{x_2+x_3}}+\sqrt{\frac{x_2+x_3}{x_3+x_4}}+\ldots+\sqrt{\frac{x_n+x_1}{x_1+x_2}}\right)-\alpha n >$$

$$> \frac{2\alpha}{\sqrt{\alpha+1}}n - \alpha n = \left(\frac{2\alpha}{\sqrt{\alpha+1}}-\alpha\right)n\,.$$

For $\alpha = \frac{5}{4}$ we have $c = 5/12$.

R e m a r k. This is a good approximation.The expression $g(\alpha) = \frac{2\alpha}{\sqrt{\alpha+1}} - \alpha$ reaches its maximal value at $\alpha = \alpha_0 \approx 1.1479$ (this is a root of the cubic equation $g'(\alpha)=0$), and the minimum value is $g(\alpha_0) \approx 0.4186$. For $\alpha = \frac{5}{4}$ we have $g(\alpha) = \frac{5}{12} \approx 0.416$.

1.10. [9]. Set $y_k = x_k + x_{k+1}$. We need to prove that

$$\frac{x_1}{y_2}+\frac{x_2}{y_3}+\ldots+\frac{x_n}{y_1} \geqslant \frac{n}{2}\,,$$

or

$$\sum_{k=1}^{n} \frac{2q_n^2 x_k - y_{k+1}}{y_{k+1}} \geqslant n(q_n^2 - 1).$$

We suppose that the parameter q_n will be chosen later. Since

$$2q_n^2 x_k - y_{k+1} = (q_n^2 x_k - x_{k+1}) + (q_n^2 x_k - x_{k+2}) \geqslant 0\,,$$

by the Cauchy-Bunyakovsky inequality for sets

$$\left\{ \sqrt{\frac{2q_n^2 x_k - y_{k+1}}{y_{k+1}}} \right\} \text{ and } \left\{ \sqrt{(2q_n^2 x_k - y_{k+1}) y_{k+1}} \right\}$$

we have

$$\sum_{k=1}^{n} \frac{2q_n^2 x_k - y_{k+1}}{y_{k+1}} \geqslant \frac{\left(\sum\limits_{k=1}^{n} (2q_n^2 x_k - y_{k+1})\right)^2}{\sum\limits_{k=1}^{n} (2q_n^2 x_k - y_{k+1}) y_{k+1}}.$$

So it suffices to prove that

$$A^2 := \left(\sum_{k=1}^{n} (2q_n^2 x_k - y_{k+1})\right)^2 \geqslant n(q_n^2 - 1) \sum_{k=1}^{n} (2q_n^2 x_k - y_{k+1}) y_{k+1} =: n(q_n^2 - 1) B.$$

Since $\sum\limits_{k=1}^{n} y_k = 2 \sum\limits_{k=1}^{n} x_k$, we have

$$A = (q_n^2 - 1) \sum_{k=1}^{n} y_k\,,$$

$$B = 2q_n^2 \sum_{k=1}^{n} x_k y_{k+1} - \sum_{k=1}^{n} y_k^2 = 2q_n^2 \sum_{k=1}^{n} y_k y_{k+1} - (q_n^2 + 1) \sum_{k=1}^{n} y_k^2\,.$$

So it remains to prove that

$$(q_n^2 - 1) \left(\sum_{k=1}^{n} y_k\right)^2 \geqslant n \left(2q_n^2 \sum_{k=1}^{n} y_k y_{k+1} - (q_n^2 + 1) \sum_{k=1}^{n} y_k^2\right). \tag{5}$$

Transform the left-hand side using the relation

$$\left(\sum_{k=1}^{n} y_k\right)^2 = n \sum_{k=1}^{n} y_k^2 - \sum_{i<k} (y_i - y_k)^2.$$

The inequality (5) will be transformed to

$$n \sum_{k=1}^{n} (y_k - y_{k+1})^2 \geqslant \left(1 - \frac{1}{q_n^2}\right) \sum_{i<k} (y_i - y_k)^2.$$

By the Cauchy-Bunyakovsky inequality

$$\sum_{k=1}^{n} (y_k - y_{k+1})^2 \geqslant \sum_{j=i}^{k-1} (y_j - y_{j+1})^2 \geqslant \frac{1}{k-j} \left(\sum_{j=i}^{k-1} (y_j - y_{j+1})\right)^2 = \frac{1}{k-j} (y_i - y_k)^2 \geqslant \frac{1}{n-1} (y_i - y_k)^2.$$

Hence

$$\frac{n(n-1)}{2} \sum_{k=1}^{n} (y_k - y_{k+1})^2 \geqslant \frac{1}{n-1} \sum_{i<k} (y_i - y_k)^2.$$

So we can take $1 - \frac{1}{q_n^2} = \frac{2}{(n-1)^2}$, i.e. $q_n = \frac{n-1}{\sqrt{n^2-2n-1}} > 1$.

Remark. When n tends to infinity, the values q_n which are found above tend to 1.

b)

1.11. (a) Denote $k_i := x_{i+1}/x_i$. Then

$$S = \frac{1}{k_1(k_2+1)} + \frac{1}{k_2(k_3+1)} + \cdots + \frac{1}{k_n(k_1+1)} \geqslant \frac{1}{a_1(a_n+1)} + \frac{1}{a_2(a_{n-1}+1)} + \cdots + \frac{1}{a_n(a_1+1)}.$$

(b) The inequality holds because

$$\frac{1}{a_i(a_{n+1-i}+1)}+\frac{1}{a_{n+1-i}(a_i+1)}=\frac{1+\frac{a_ia_{n+1-i}-1}{(1+a_i)(1+a_{n+1-i})}}{a_ia_{n+1-i}}\geqslant b_i$$

where the latter inequality holds because $(1+a_i)(1+a_{n+1-i})\geqslant(1+\sqrt{a_ia_{n+1-i}})^2$.

(c) The first inequality $2S\geqslant g(\ln(a_1a_n))+g(\ln(a_2a_{n-1}))+\cdots+g(\ln(a_na_1))$ holds because $g(x)$ is less than both e^{-x} and $2(e^x+e^{x/2})^{-1}$. The second inequality holds by the Jensen inequality because g is convex.

(d) [Dr]

2.1. a) [20]

For $n=4$ we need to prove that

$$(x_1+x_2+x_3+x_4)^2\geqslant 2x_1x_2+2x_2x_3+2x_3x_4+2x_4x_1+4x_1x_3+4x_2x_4\,.$$

This follows from the inequality

$$x_1^2+x_2^2+x_3^2+x_4^2\geqslant 2x_1x_3+2x_2x_4\,.$$

For $n=3$ and $n=5$ re-write the inequality. We need to prove that

$$(n-1)(a_1+a_2+\ldots+a_n)^2\geqslant 2n\sum_{i<k}a_ia_k. \tag{6}$$

Indeed, notice that the Cauchy–Bunyakovsky inequality applied to sets $a_1,a_2,\ldots,a_n$ and $1,1,\ldots,1$ gives us:

$$n(a_1^2+a_2^2+\ldots+a_n^2)\geqslant(a_1+a_2+\ldots+a_n)^2.$$

Now we have

$$n(a_1+a_2+\ldots+a_n)^2=n(a_1^2+a_2^2+\ldots+a_n^2)+2n\sum_{i<k}a_ia_k\geqslant(a_1+a_2+\ldots+a_n)^2+2n\sum_{i<k}a_ia_k\,,$$

which implies (6).

Now assume that $n\geqslant 6$. We may suppose that $x_3\geqslant x_1$ and $x_3\geqslant x_2$ (e.g. make a cyclic shift of variables such that x_3 becomes the maximum). For $r=1$, 2, and 3 denote by a_r the sum of all x_k such that $k\equiv r\pmod 3$ and $k\leqslant n$. Then $x_1+x_2+\ldots+x_n=a_1+a_2+a_3$. Hence by (6) we have

$$(x_1+x_2+\ldots+x_n)^2=(a_1+a_2+a_3)^2\geqslant 3(a_1a_2+a_2a_3+a_3a_1)=3\cdot\sum_{(i-k)\not\vdots 3}x_ix_k\,.$$

Set

$$A:=\sum_{(i-k)\not\vdots 3}x_ix_k\quad\text{and}\quad B:=\sum_{k=1}^{n}x_k(x_{k+1}+x_{k+2}).$$

We have $A\geqslant B$ because

- for $n\equiv 0\pmod 3$ all the summands of B are contained in A;
- for $n\equiv 1\pmod 3$ the sum A contains all the summands of B except x_nx_1, but x_nx_1 does not exceed x_nx_3;
- for $n\equiv 2\pmod 3$ the sum A contains all the summands of B except $x_{n-1}x_1$ and x_nx_2, but these summands do not exceed $x_{n-1}x_3$ and x_nx_3, respectively.

Hence

$$(x_1+x_2+\ldots+x_n)^2\geqslant 3A\geqslant 3B=3\sum_{k=1}^{n}x_k(x_{k+1}+x_{k+2}).$$

In order to show that $\min\left\{\frac{n}{2},3\right\}$ is the sharp constant for $n\leqslant 6$ we set $x_1=x_2=\ldots=x_n=1$ and for $n\geqslant 6$ we set $x_1=x_2=x_3=1$ and $x_4=x_5=\ldots=x_n=0$.

b) The case $n<6$ is trivial. For $n=6$ the equality is achieved when $x_1+x_4=x_2+x_5=x_3+x_6$. For $n\geqslant 6$ the equality is achieved for the sets of form $(t,1,1,1-t,0,\ldots,0)$, where $t\in[0,1]$, and their cyclic shifts.

2.2. [20]

For $n=4$ and $n=7$ this is a particular case of (6).

For $n=5$ the inequality coincides with $\sum(x_k-2x_{k+2}+x_{k+4})^2\geqslant 0$.

For $n=6$ the inequality follows from $x_1^2+x_2^2+\ldots+x_6^2\geqslant 2x_1x_4+2x_2x_5+2x_3x_6$.

For $n=8$ open brackets in the following corollary of the Cauchy–Bunyakovsky inequality

$$4(x_1^2+x_2^2+x_3^2+x_4^2)\geqslant(x_1+x_2+x_3+x_4)^2.$$

We obtain

$$3(x_1^2+x_2^2+x_3^2+x_4^2)\geqslant 2(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4).$$

Hence

$$3(x_1+x_2+x_3+x_4)^2 \geqslant 8(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4), \tag{7}$$

This is the required inequality for $n=8$.

Now assume that $n>8$. We may suppose that $x_4 \geqslant x_1$, $x_4 \geqslant x_2$, and $x_4 \geqslant x_3$. For $r=1$, 2, 3, and 4 denote by a_r the sum of all x_k such that $k \equiv r \pmod 4$ and $k \leqslant n$. Then $x_1+x_2+\ldots+x_n = a_1+a_2+a_3+a_4$. Hence by (7)

$$3(x_1+x_2+\ldots+x_n)^2 = 3(a_1+a_2+a_3+a_4)^2 \geqslant 8(a_1a_2+a_2a_3+a_3a_4+a_4a_1) \geqslant 8\cdot \sum_{(i-k)\not\vdots 4} x_ix_k\,.$$

Set

$$A := \sum_{(i-k)\not\vdots 4} x_ix_k \quad \text{and} \quad B := \sum_{k=1}^{n} x_k(x_{k+1}+x_{k+2}+x_{k+3}).$$

We have $A \geqslant B$ because

- for $n \equiv 0 \pmod 4$ all the summands of B are contained in A;
- for $n \equiv 1 \pmod 4$ the sum A contains all the summands of B except x_nx_1, but x_nx_1 does not exceed x_nx_4;
- for $n \equiv 2 \pmod 4$ the sum A contains all the summands of B except $x_{n-1}x_1$ and x_nx_2, but these summands do not exceed $x_{n-1}x_4$ and x_nx_4;
- for $n \equiv 3 \pmod 4$ the sum A contains all the summands of B except $x_{n-2}x_1$, $x_{n-1}x_2$ and x_nx_3, but these summands do not exceed $x_{n-2}x_4$, $x_{n-1}x_4$, and x_nx_4.

Hence

$$3(x_1+x_2+\ldots+x_n)^2 \geqslant 8A \geqslant 8B = 8\sum_{k=1}^{n} x_k(x_{k+1}+x_{k+2}+x_{k+3}).$$

2.3. a) Cf. [11]. By the Cauchy–Bunyakovsky inequality and Problem 2.2 we have

$$\frac{x_1}{x_2+x_3+x_4}+\frac{x_2}{x_3+x_4+x_5}+\ldots+\frac{x_{n-1}}{x_n+x_1+x_2}+\frac{x_n}{x_1+x_2+x_3} \geqslant \frac{(x_1+x_2+\ldots+x_n)^2}{\sum x_k(x_{k+1}+x_{k+2}+x_{k+3})} \geqslant \frac{n}{3}.$$

b) ???

2.4. [1, Problem 187]. We may assume that $x_1 \leqslant x_2$. Set

$$S := x_1+x_2+\ldots+x_n, \quad S_1 := x_1+x_3+\ldots, \quad S_2 := x_2+x_4+\ldots.$$

Then $S_1^2+S_2^2 \geqslant (S_1+S_2)^2/2 = S^2/2$. Hence

$$\frac{S^2}{2} \geqslant S^2 - S_1^2 - S_2^2 = 2\sum_{(i-k)\not\vdots 2} x_ix_k. \tag{8}$$

If n is even, then the last sum contains all the summands of form x_kx_{k+1}. If n is odd, then the summand x_nx_1 is missing, however the sum contains a greater summand x_nx_2. So

$$\frac{S^2}{2} \geqslant 2(x_1x_2+x_2x_3+\ldots+x_nx_1).$$

2.5. See the solution of 1.3 up to the inequality (4).

2.6. Induction on $n \geqslant n=4$. Denote the left-hand side by L_n. We have

$$L_4 = \frac{x_1+x_3}{x_2+x_4}+\frac{x_2+x_4}{x_1+x_3} = a+a^{-1} \geqslant 2.$$

Let us prove the inductive step. We may assume that x_{n+1} is the minimal of all x_i's. Now remove the last summand from L_{n+1}, and then decrease two others. We obtain

$$L_{n+1} \geqslant \frac{x_1}{x_{n+1}+x_2}+\ldots+\frac{x_n}{x_{n-1}+x_{n+1}} \geqslant \frac{x_1}{x_n+x_2}+\ldots+\frac{x_n}{x_{n-1}+x_n} = L_n\,.$$

In order to show that the constant 2 is sharp, take

$$x_1=x_2=1, \quad x_3=t, \quad x_4=t^2, \quad \ldots, \quad x_n=t^{n-2}\,.$$

When $t \to +0$, the first two summands tend to 1 and the remaining tends to 0.

2.7. [10]. Set $S := x_1 + x_2 + \ldots + x_n$. Use the Cauchy–Bunyakovsky inequality for sets $\left\{\frac{x_k+x_{k+1}}{x_k+x_{k+2}}\right\}$ and $\{(x_k + x_{k+1})(x_k + x_{k+2})\}$. We obtain

$$\frac{x_1+x_2}{x_1+x_3} + \frac{x_2+x_3}{x_2+x_4} + \ldots + \frac{x_{n-1}+x_n}{x_{n-1}+x_1} + \frac{x_n+x_1}{x_n+x_2} \geqslant \frac{4(x_1+x_2+\ldots+x_n)^2}{\sum\limits_{k=1}^{n}(x_k+x_{k+1})(x_k+x_{k+2})}.$$

So it suffices to prove that

$$S^2 \geqslant \sum_{k=1}^{n}(x_k+x_{k+1})(x_k+x_{k+2}) = \sum_{k=1}^{n} x_k^2 + 2\sum_{k=1}^{n} x_k x_{k+1} + \sum_{k=1}^{n} x_k x_{k+2}.$$

This can be shown by opening brackets in the left-hand side, because for $n \geqslant 4$ all the summands $x_k x_{k+1}$ and $x_k x_{k+2}$, where $k = 1, 2, \ldots, n$, are different.

In order to show that the constant 4 is sharp, take $x_k = a^{k-1}$ for $k = 1, 2, \ldots, n-1$ and $x_n = a^{n-2}$. When $a \to \infty$, the first $n-3$ summands tend to 0 and the remaining summands tend to 1, 2 and 1.

Using the Cauchy–Bunyakovsky inequality as it is done in the solution of the next problem, the reader will easily find another solution of this problem reducing it to the inequality from Problem 2.4.

2.8. [6]. Use the Cauchy–Bunyakovsky inequality for sets $\left\{\frac{x_k}{x_{k-1}+x_{k+2}}\right\}$ пїЅ $\{x_k(x_{k-1}+x_{k+2})\}$. We obtain

$$\frac{x_1}{x_n+x_3} + \frac{x_2}{x_1+x_4} + \ldots + \frac{x_{n-1}}{x_{n-2}+x_1} + \frac{x_n}{x_{n-1}+x_2} \geqslant \frac{(x_1+x_2+\ldots+x_n)^2}{(x_1x_2+x_2x_3+\ldots+x_nx_1)+(x_1x_3+x_2x_4+\ldots+x_nx_2)}.$$

So it suffices to prove that

$$S^2 \geqslant 3(x_1x_2+x_2x_3+\ldots+x_nx_1)+3(x_1x_3+x_2x_4+\ldots+x_nx_2) =: 3Y,$$

where $S := x_1 + x_2 + \ldots + x_n$. Set

$$S_1 := x_1 + x_4 + \ldots, \quad S_2 = x_2 + x_5 + \ldots \quad \text{and} \quad S_3 = x_3 + x_6 + \ldots.$$

Then $S = S_1 + S_2 + S_3$ and $S_1^2 + S_2^2 + S_3^2 \geqslant S^2/3$. We may assume that $x_3 \geqslant x_1$ and $x_3 \geqslant x_2$. Notice that

$$S^2 \geqslant \frac{3}{2}(S^2 - S_1^2 - S_2^2 - S_3^2) = 3\sum_{(i-k)\not\vdots 3} x_i x_k =: 3Z.$$

- If $n \equiv 0 \pmod 3$, then all the summands of Y are contained in Z.
- If $n \equiv 1 \pmod 3$, then Z contains all the summands of Y except $x_n x_1$, but this summand does not exceed $x_n x_3$.
- If $n \equiv 2 \pmod 3$, then Z contains all the summands of Y except $x_{n-1}x_1$ and $x_n x_2$, but these summands do not exceed $x_{n-1}x_3$ and $x_n x_3$.

Hence $S^2 \geqslant 3Y \geqslant 3Z$, which proves the initial inequality.

In order to show that the constant 3 is sharp, take $x_k = a^{k-1}$ for $k = 1, 2, \ldots, n-2$ and $x_{n-1} = x_n = 1$. When $a \to 0$, the first and the last two summands tend to 1, while the remaining summands tend to 0.

2.9. [5]. The inequality is obtained by summing two inequalities of 2.8 (for the direct and the opposite order of variables).

In order to show that the constant 6 is sharp, take $x_k = a^{k-1}$ for $k = 1, 2, \ldots, n-2$ and $x_{n-1} = x_n = 1$. When $a \to 0$, the last four summands tend to 1, 2, 2, 1, respectively; the remaining tend to 0.

2.10. This is conjectured in [19].

The following proof is due to P.Milošević пїЅ M. Bukić, participants of the Conference.

This inequality can be represented as sum of two inequalities for $n = 2004$ — the inequality from Problem 2.8 and the inequality

$$\frac{x_1}{x_1+x_4} + \frac{x_2}{x_2+x_5} + \ldots + \frac{x_n}{x_n+x_3} \geqslant 3\,.$$

Prove the last inequality. For $n = 3m$ it is the sum of three inequalities:

$$\frac{x_1}{x_1+x_4} + \frac{x_4}{x_4+x_7} + \ldots + \frac{x_{n-2}}{x_{n-2}+x_1} \geqslant 1\,.$$
$$\frac{x_2}{x_2+x_5} + \frac{x_5}{x_5+x_8} + \ldots + \frac{x_{n-1}}{x_{n-1}+x_2} \geqslant 1\,.$$
$$\frac{x_3}{x_3+x_6} + \frac{x_6}{x_6+x_9} + \ldots + \frac{x_n}{x_n+x_3} \geqslant 1\,.$$

Each of these inequalities can be re-written as

$$\frac{1}{1+a_1}+\frac{1}{1+a_3}+\ldots+\frac{1}{1+a_m}\geqslant 1 \qquad \text{where } \ a_1a_2\ldots a_m=1.$$

This can be shown by induction. The base $m=2$ is the following inequality:

$$\frac{1}{1+a_1}+\frac{1}{1+\frac{1}{a_1}}=1\geqslant 1\,.$$

To prove the induction step, let us check that

$$\frac{1}{1+b}+\frac{1}{1+c}\geqslant\frac{1}{1+bc}\,.$$

This can be done directly by reducing to a common denominator and opening brackets.

Here is the proof of A. Khrabrov. Let us prove that

$$Z:=\frac{x_1+x_2}{x_1+x_4}+\frac{x_2+x_3}{x_2+x_5}+\ldots+\frac{x_{3n}+x_1}{x_{3n}+x_3}\geqslant 6.$$

Set $x_{3n+k}:=x_k$ and, for $r=0,1,2$,

$$S_r:=\sum_{k=1}^n x_{3k+r}, \quad X_r:=\sum_{k=1}^n\frac{x_{3k+r}}{x_{3k+r}+x_{3k+3+r}}, \quad \text{and} \quad Y_r:=\sum_{k=1}^n\frac{x_{3k+r+1}}{x_{3k+r}+x_{3k+3+r}}.$$

First we prove that $X_r\geqslant 1$. Consider only the case $r=0$. Then

$$X_0S_0^2\geqslant X_0\left(\sum_{k=1}^n x_{3k}^2+\sum_{k=1}^n x_{3k}x_{3k+3}\right)=X_0\left(\sum_{k=1}^n x_{3k}(x_{3k}+x_{3k+3})\right)\geqslant S_0^2,$$

where the last inequality holds by the Cauchy–Bunyakovsky inequality. So $X_0\geqslant 1$.

Now prove that $Y_r\geqslant S_{r+1}/S_r$ (we set $S_3:=S_0$). Consider only the case $r=0$.

$$Y_0S_0S_1\geqslant Y_0\left(\sum_{k=1}^n x_{3k}x_{3k+1}+\sum_{k=1}^n x_{3k+1}x_{3k+3}\right)=Y_0\left(\sum_{k=1}^n x_{3k+1}(x_{3k}+x_{3k+3})\right)\geqslant S_1^2,$$

where the last inequality holds by the Cauchy–Bunyakovsky inequality. So $Y_0\geqslant S_1/S_0$.

Summing up all the proved inequalities we obtain

$$Z=X_0+X_1+X_2+Y_0+Y_1+Y_2\geqslant 3+\frac{S_1}{S_0}+\frac{S_2}{S_1}+\frac{S_0}{S_2}\geqslant 6.$$

In order to show that the constant 6 is sharp, take $x_1=x_2=x_3=1$, $x_k=a^{n-k+1}$ for $k=3,\,4,\,\ldots,\,n$. When $a\to 0$, the first and the second summands tend to 2, the third and the last tend to 1, and the remaining summands tend to 0.

2.11. This proof is due to A. Khrabrov. Set $S=x_1+x_2+\ldots+x_n$ and $T=\sum\limits_{(i-k)\not\,\vdots\, 2}x_ix_k$. By the Cauchy–Bunyakovsky inequality for sets $\left\{\dfrac{x_k}{x_{k-1}+x_{k+3}}\right\}$ and $\{x_k(x_{k-1}+x_{k+3})\}$ we have

$$\frac{x_1}{x_n+x_4}+\frac{x_2}{x_1+x_5}+\ldots+\frac{x_{n-1}}{x_{n-2}+x_2}+\frac{x_n}{x_{n-1}+x_3}\geqslant\frac{(x_1+x_2+\ldots+x_n)^2}{(x_1x_2+x_2x_3+\ldots+x_nx_1)+(x_1x_4+x_2x_5+\ldots+x_nx_3)}.$$

So it suffices to prove that

$$S^2\geqslant 4(x_1x_2+x_2x_3+\ldots+x_nx_1)+4(x_1x_4+x_2x_5+\ldots+x_nx_3).$$

In the solution of problem 2.4 we proved that $S^2\geqslant 4T$, see (8). So it suffices to prove that

$$T\geqslant(x_1x_2+x_2x_3+\ldots+x_nx_1)+(x_1x_4+x_2x_5+\ldots+x_nx_3). \tag{9}$$

Since n is even, all the summands of the right-hand sum are contained in the left-hand sum.

In order to show that the constant 6 is sharp, take $x_k=a^{k-1}$ and $k=1,\,2,\,\ldots,\,n-3$ and $x_{n-2}=x_{n-1}=x_n=1$. When $a\to +0$ the first summand and the three last summands tend to 1, and the remaining summands tend to 0.

2.12. [14]. Note that $a^2 - ab + b^2 \leqslant \max\{a, b\}^2$.

Let x_{i_1} be the maximal number of $x_1, x_2, \ldots, x_n$. Let x_{i_2} be the maximal number of x_{i_1+1} and x_{i_1+2}. Let x_{i_3} be the maximal number of x_{i_2+1} and x_{i_2+2}, and so on. There exists a number k such that $x_{i_{k+1}} = x_{i_1}$. Hence

$$\sum_{k=1}^{n} \frac{x_k^2}{x_{k+1}^2 - x_{k+1}x_{k+2} + x_{k+2}^2} \geqslant \sum_{j=1}^{k} \frac{x_{i_j}^2}{x_{i_{j+1}}^2} \geqslant k \geqslant \left[\frac{n+1}{2}\right],$$

where the latter inequality holds because $k \geqslant n/2$.

In order to show that the constant $\left[\frac{n+1}{2}\right]$ is sharp, take $x_k = 1$ for odd k and $x_k = 0$ for even k. Then the left-hand side is $\left[\frac{n+1}{2}\right]$.

References

[1] *Васильев Н. Б., Егоров А. А.* Задачи Всесоюзных математических олимпиад. М.: Наука, 1988.

[2] *Дринфельд В. Г.* Об одном циклическом неравенстве // Мат. заметки. 1971. Т. 9. № 2. С. 113–119.

[3] *Курляндчик Л. Д., Файбусович А.* История одного неравенства // Квант. 1991. № 4. С. 14–18.

[4] *Толпыго А. К.* Тысяча задач Международного математического Турнира городов. М.: МЦНМО, 2009.

[5] *Чимэдцэрэн С.* Нэгэн орчилт нийлбэр // Математикийн олимпиадын цуврал. 1999. Т. 22. (На монгольск. яз.).

[6] *Чимэдцэрэн С., Адъяасурен В., Батболд С.* Оценка в одной циклической сумме // Монгол улсын их сургууль, Эрдэм шинжилгээний бичиг. 2000. Т. 7 (168). С. 79–84.

[7] *Bushell P. J.* Shapiro's Cyclic Sum // Bull. London Math. Soc. 1994. Vol. 26. No 6. P. 564–574

[8] *Bushell P. J., McLeod J. B.* Shapiro's cyclic inequality for even n // J. Inequal. & Appl., 2002. Vol. 7(3). P. 331–348

[9] *Cîrtoaje V.* Crux Mathematicorum. 2006. Vol. 32. No. 8. Problem 3195.

[10] *Daykin D. E.* Inequalities for certain cyclic sums // Proc. Edinburgh Math. Soc. (2) 1970/71. Vol. 17. P. 257–262.

[11] *Diananda P. H.* Extensions of an inequality of H. S. Shapiro // Amer. Math. Monthly 1959. Vol. 66. P. 489–491.

[12] *Diananda P. H.* On a conjecture of L. J. Mordell regarding an inequality involoving quadratic forms // J. London Math. Soc. 1961. Vol. 36. P. 185–192.

[13] *Diananda P. H.* Inequalities for a class of cyclic and other sums // J. London Math. Soc. 1962. Vol. 37. P. 424–431.

[14] *Diananda P. H.* Some cyclic and other inequalities // Proc. Cambridge Philos. Soc. 1962. Vol. 58. P. 425–427.

[15] *Diananda P. H.* Some cyclic and other inequalities, II // Proc. Cambridge Philos. Soc. 1962. Vol. 58. P. 703–705.

[16] *Diananda P. H.* On a cyclic sum // Proc. Glasgow Math. Assoc. 1963. Vol. 6. P. 11–13.

[17] *Elbert A.* On a cyclic inequality // Period. Math. Hungarica. 1973. Vol. 4. № 2–3. P. 163–168.

[18] *Malcolm M. A.* A note on a conjecture of L. J. Mordell // Math. Comp. 1971. Vol. 25. P. 375–377.

[19] *Mitrinović D. S., Pečarić J. E., Fink A. M.* Classical and new inequalities in analysis. Kluwer Academic Publishers Group, Dordrecht, 1993. (Mathematics and its Applications (East European Series), Vol. 61).

[20] *Mordell L. J.* On the inequality $\sum\limits_{r=1}^{n} \frac{x_r}{x_{r+1}+x_{r+2}} \geqslant \frac{n}{2}$ and some others // Abh. Math. Sem. Univ. Hamburg. 1958. Vol. 22. P. 229–240.

[21] *Nowosad P.* Isoperimetric eigenvalue problems in algebras // Comm. Pure Appl. Math. 1968. Vol. 21. P. 401–465.

[22] *Shapiro H. S., Northover F. H.* Amer. Math. Monthly. 1956. Vol. 63. № 3. P. 191–192.

[23] *Tanahashi K., Tomiyama J.* Indecomposable positive maps in matrix algebras // Canad. Math. Bull. 1988. Vol. 31. № 3. P. 308–317.

[24] *Troesch B. A.* Full solution of Shapiro's cyclic inequality // Notices Amer. Math. Soc. 1985. Vol. 39. № 4. P. 318.

[25] *Vukmirović J.* A note on an inequality for the cyclic sums introduced by D. E. Daykin // Math. Balk. 1978. Vol. 8. P. 293–297.

[26] *Yamagami S.* Cyclic inequalities // Proc. Amer. Math. Soc. 1993. Vol. 118. № 2. P. 521–527.

[27] *Zulauf A.* Note on a conjecture of L. J. Mordell // Abh. Math. Sem. Univ. Hamburg. 1958. Vol. 22. P. 240–241.

[28] *Zulauf A.* Note on an Inequality // Math. Gazette. 1962. Vol. 46. №355. P. 41–42.