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Colorings and clusters

A. Belov-Kanel,

I. Ivanov-Pogodaev, A. Malistov, M. Kharitonov

1. Each point of the plane is colored by one of **a)** two colors; **b)** three colors. Prove that there exist two points separated by the distance 1 which have the same color.

2. Prove the same for the space colored by 4 colors.

3. Prove the same for the n-dimensional space colored by $n+1$ colors.

4. The plane is colored by two colors. Prove that one can choose the color such that for any distance d there exist the points separated by the distance d and colored by this color.

5. Prove the same for the n-dimensional space colored by n colors.

6.* Prove the same for the plane colored by 3 colors.

7.* Prove the same for the n-dimensional space colored by $n+1$ colors.

8. Color the plane such that there are no unit segment with the edges colored by the same color. Try to use the minimal number of colors.

So far, it remains an open question to find the minimal number of colors x such for some coloring there are no unit segment with the single-colored edges. It is known that $4 \leq x \leq 7$.

If we look for "almost unit" segments then the problem became more simple.

9. The plane is colored by **a)** four colors; **b)** five colors. Prove that there exist two points of the same color such that the distance between them is different of 1 by less then 0.001.

The solution of the b-part of the previous problem is based on the following fact:

10. Consider the cell-like plane. Suppose that any cell is fully colored by one of two colors. Let the common edge of the two cells be colored by both cell-colors. Prove that there exists a single-colored polygonal path such that the edges of the path are separated for the distance more then 1000.

11. Prove the generalization of the previous problem for 3-dimensional cube lattice colored by 3 colors.

12. Prove the generalization of the previous problem for n-dimensional cube lattice colored by n colors.

13. Suppose that a cube $k \times k \times k$ consists of k^3 unit cubes which are colored by red, blue and green colors. Prove that there exists a single-colored polygonal path which connects opposite faces of the big cube. Formulate and prove the n-dimensional generalization. Prove that if we use $n+1$ colors then the statement is not true.

14. 3-dimensional space is colored by 9 colors. Prove that there exist two single-colored points which are separated by the distance d such that $|d-1| < 0.001$. Generalize this problem for n-dimensional case.

Definition. Connected set of cells called *claster* (Two cells with common point are *connected*).

The achievements of the problem 12 can be further generalized.

15. Suppose that the cells of the n-dimensional lattice are colored by k colors and $k < n+1$. Then in any cube with edge $10M$ there exist connected single-colored cluster with volume M^{n+1-k}.

a)* Solve the problem for $k = 2$.

b)* Solve the problem for $k = n$.

c)** Try to solve the problem for other cases

16. Show that the statement of the problem 12 can be obtained from the following fact, which is base for topological definition of dimension: if n-dimensional space is covered by some sets of bounded diameter then there exists a point covered $n+1$ times.

17.* Prove this topological fact.