On isogonal conjugacy, Miquel points, (Newton-)Gauss lines, etc.

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1 Introductory problems

1. **Isogonal conjugacy.** Given a triangle ABC and a point P.
a)Prove that lines symmetric to AP, BP, CP in the bisectors of corresponding angles are concurrent or parallel. The common point P' of these lines is called *isogonal conjugate* to P with respect to ABC.
b)Prove that P' is a point at infinity (i.e. three corresponding lines are parallel) iff P lies on the circumcircle of ABC.
c)Determine the image isogonal conjugacy of a circle passing through two of three points A, B, C.
d)[1] Prove that all projections of P and P' to the sidelines of ABC are concyclic. Reformulate the statement above for the case when P' is a point at infinity.
e)For two pairs X, X' and Y, Y' of isogonal conjugate points, prove that $XY \cap X'Y'$ and $XY' \cap X'Y$ are isogonal conjugates.

Given a quadrilateral $ABCD$ and a point P.
f)Suppose that three of four lines symmetric to AP, BP, CP, DP in the bisectors of corresponding angles are concurrent. Prove that all four lines are concurrent.
g)Prove that for a point P there exists an isogonal conjugate P' iff projections of P to the sidelines of $ABCD$ are concyclic (if P' exists, then all projections of P and P' to the sidelines of $ABCD$ are concyclic).

A conic is said to be inscribed to a polygon if it touches all the sidelines of this polygon.
h)Prove that foci of a conic inscribed to a triangle are isogonal conjugates.
i)Prove that focus of a parabola inscribed to a triangle lies on its circumcircle.

2. **Miquel point.** Given a quadrilateral $ABCD$. Let $E = AB \cap CD$, $F = AD \cap BC$.
a)Prove that (in notation of the previous problem) circumcircles of triangles ABF, CDF, ADE, CDE have a common point M (Miquel point for a quadruple of lines AB, BC, CD, DA).
b)Prove that M is a center of spiral similarity that takes segment BE to FD (or DE to FB, etc.)
c)Two bugs B and C move, each at a constant speed, along two lines intersecting at A. Prove that all the circles ABC have a common point, and * all the lines BC touch fixed parabola.
d)(IMO2005) Let $ABCD$ be a convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all triangles PQR as E and F vary. Prove that the circumcircles of these triangles have a common point other than P.
e)Establish a connection between Miquel point and inscribed conics.
f)Prove that the projections of Miquel point to the sidelines of a quadrilateral lie on a line perpendicular to Gauss line. Establish a connection between this line and a parabola inscribed to the quadrilateral.

3. **Gauss line.** Given a quadrilateral $ABCD$. Let $E = AB \cap CD$, $F = AD \cap BC$.
a)Prove that the midpoints of the segments AC, BD, EF lie on a line (that is called Gauss line of $ABCD$, or Gauss line of quadruple of lines AB, BC, CD, DA).
b)Prove that the center of the circle passing through the projections of a pair of isogonal conjugates lies on Gauss line.
c)Prove that Miquel point is isogonal conjugate to the infinite point of Gauss line.
d)Prove that centers of conics inscribed to a quadrilateral lie on Gauss line.

[1]Here and further we footnotesize the statements that not used in the proofs of results from section 2.

e)(All-Russian Olympiad 2009) Let A_1 and C_1 be points on the sides AB and BC of parallelogram $ABCD$. Let $P = AC_1 \cap CA_1$. Circumcircles of triangles AA_1P and CC_1P meet for the second time at point Q lying inside triangle ACD. Prove that $\angle PDA = \angle QBA$.

2 Three Miquels for a Quartet.

In this section we use the following notation. Let A, B, C, D be four points such that no three of them are collinear. Let X be Miquel point for the quadruple of lines AB, AC, BD, CD, let Y be Miquel point for the quadruple of lines AB, AD, BC, CD, let Z be Miquel point for the quadruple of lines BC, AC, BD, AD. We set $P_X = AD \cap BC$, $P_Y = AC \cap BD$, $P_Z = AB \cap CD$. Let K_X and L_X be midpoints of the segments BC and AD respectively, similarly, let K_Y, L_Y be midpoints of AC, BD, let K_Z, L_Z be midpoints of AB, CD. Let $\Gamma_X = K_X L_X$, $\Gamma_Y = K_Y L_Y$, $\Gamma_Z = K_Z L_Z$ be Gauss lines for the corresponding quadruples of lines.

4. Prove that AX, BY, CZ have a common point D', or parallel. Similarly define A', B', C'.

5. Prove that A', B', C', D' are isogonal conjugates to A, B, C, D with respect to triangle XYZ.

6. Prove that X, Y, Z are Miquel points for quadruples of lines joining A', B', C', D'.

7. Probe that lines AA', BB', CC', DD' are parallel.

8. Prove that AD, $A'D'$, YZ are concurrent (find other analogous intersections).

9.

a)Prove that points X, Z, P_Y, K_Y, K_Y lie on a certain circle ω_Y. Similarly define circles ω_X, ω_Z.

b)Prove that ω_X, ω_Y, ω_Z have a common point T.

c)Prove that XP_X, YP_Y, ZP_Z meet at T.

3 Quartets for three Miquels.

Let XYZ be a triangle. Define a transformation ψ_X as the symmetry in the bisector of angle X followed by the inversion with center X and radius $R = \sqrt{XY \cdot XZ}$. Similarly define transformations ψ_Y, ψ_Z.

10. Prove that

a)$\psi_X(Y) = Z$, $\psi_X(Z) = Y$;

b)ψ_X^2 is the identity transformation;

c)Product $\psi_Z\psi_Y\psi_X$ is the identity transformation.

Let D be an arbitrary point, let $A = \psi_X(D)$, $B = \psi_Y(D)$, $C = \psi_Z(D)$.

11. Prove that $\triangle XDZ \sim \triangle XYA$ and $\triangle XDY \sim \triangle XZA$.

12. Prove that each of the transformations ψ_X, ψ_Y, ψ_Z takes the 4-element set $\{A, B, C, D\}$ to itself. A 4-element set of points $\{A, B, C, D\}$ defined as above is said to be a *quartet*. From the previous problem it follows that all the plane except X, Y, Z is partitioned into quartets.

13. Prove that four isogonal conjugates to points of a quartet is a quartet.

14. Find all the quartets containing

a)the incenter I of triangle XYZ;

b)the circumcenter O of triangle XYZ.

c)Find the invariant points for ψ_Z, and corresponding quartets.

15.

a)Prove that X is Miquel point for the quadruple of lines AB, AC, BD, CD.

b)Formulate similar statements for Y, Z.

c)Prove the converse: if X, Y, Z are Miquel points defined by A, B, C, D, then A, B, C, D us a quartet (for X, Y, Z).

16. Prove that each of transformations ψ_X, ψ_Y, ψ_Z commutes with the isogonal conjugacy with respect to XYZ.

17. Suppose A, B, C, D be a quartet with respect to XYZ, let A', B', C', D' be isogonal conjugates to A, B, C, D respectively. Consider four conics having pairs of foci A and A', B and B', C and C', D and D'.

a)Prove that these conics are homothetic to each other.

b)Prove that midpoints of six segments joining centers of these conics lie on a certain conic that is homothetic to them and passing through X, Y, Z.

18. Let M, N be a pair of isogonal conjugates with respect to triangle ABC lying inside ABC. It appears that $AM \cdot AN \cdot BC = BM \cdot BN \cdot AC = CM \cdot CN \cdot AB = k$.

a)Prove that the midpoint of MN is the gravity center of A, B, C.

b)Find k in terms of side lengths of ABC.

4 Additional problems.

19.
a)Let A, B, C, D be a quartet, A', B', C', D' be conjugated quartet; let P_X be intersection point of AD and BC, P_Y — of AC and BD, P_Z — of AB and CD. Points Q_X, Q_Y, Q_Z are defined similarly by points A', B', C', D'. Prove that lines P_XQ_X, P_YQ_Y, P_ZQ_Z are concurent in the point, which lie on the circumcircle of triangle XYZ (notations as above).
b)In previous notations prove that lines P_XQ_Y, P_YQ_X and XY concur.
c)Let Z' be the point obtained in b). Prove that line ZZ' is parallel to AA', BB', CC', DD'.
d)Let D_1, D_1' and D_2, D_2' be two pairs of isogonally conjugated points such that $D_1D_1' \parallel D_2D_2'$. Prove that lines $A_1A_2, B_1B_2, C_1C_2, D_1D_2$ concur (A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2 are quartets).
20. Given points A, B, C, D. It is known that triangle XYZ is perspective to each of triangles ABC, BCD, CDA, DAB (with indicated order of vertices). Points D', A', B', C' are respective centers of perspective. Prove that lines AA', BB', CC', DD' concur.