Excircles and Dozens of Points.

Hints, Solutions, Comments.

Series A: The First Dozen: Touch Points

A1. Follows from the calculation of the tangent segments, for example, $2AB_1 = AB_1 + AC_1 = AB + BA_0 + AC + CA_0 = 2p$, hence $B'B_1 = p - \frac{b}{2} = \frac{a+c}{2}$. (Also see a comment on B5)

A2. Follows from Ceva Theorem using the equality of the segments of tangents).

A3. From A1 it follows that A' equal powers with respect to the circles ω_2 and ω_3, hence A' lies on the radical axis of these circles. This radical axis is perpendicular to I_2I_3, hence it is parallel to the bisector of the angle BAC (or $B'A'C'$). Thus this radical axis as a bisector of the angle $B'A'C'$. Hence radical centers of the triples of circles are I_0', I_1', I_2', I_3'.

A4#. (This Problem was formulated in thesis of K. Kuznetsova (Velikie Luki) at the Conference "Start v Nauku — 2009")

From A3 it follows that there exists an inversion with center I_0' that takes each of the circles ω_1, ω_2, ω_3 to itself. This inversion takes AB, BC, CA to the circles passing through I_0' and touching ω_1, ω_2, ω_3.

A5#. The homothety with center A taking ω_1 to ω_0 takes diameter KA_1 to diameter A_0L. Thus AA_1 coincides to LA_1. I_0A' is a midline of the triangle A_0LA_1. This completes the solution.

A6#. Using the notation of the previous solution: triangles A_1LA_0 and A_1AH_a are homothetic (with center A_1), hence A_1I_0 is the median in triangle A_1AH_a passing through the midpoint of the altitude AH_a.

Series B: The Second Dozen: "Foci"

B0. The statement follows since A_iB_i is parallel to a bisector (either internal or external) of angle C.

B1. From B0 it follows that $A_2B_{23} \perp A_3B_{23}$ и $A_2C_{23} \perp A_3C_{23}$, hence 4 mentioned points lie on the circle with diameter A_2A_3. From A1 it follows that the center of this circle is A' (and radius equals to $\frac{b+c}{2}$. Similarly, points B_{01}, C_{01}, A_0, A_1 lie on the circle with center A' (and radius equals $\frac{|b-c|}{2}$.

B2. In a right-angled triangle $A_2B_{23}A_3$: $A'B_{23} = A'A_2$, $(= \frac{b+c}{2}$ — see A2), hence equilateral triangles $A_2A'B_{23}$ and A_2CB_2 are homothetic, and $A'B_{23} \parallel AC$, that means that B_{23} lies on the midline $A'C'$.

Note. B_{23} also lies on the circle with diameter B_2B_3.

B3. Determine the length $A'B_{23}$ (see B1), and obtain $C'B_{23} = A'B_{23} = A'C' = \frac{c}{2}$, hence B_{23} lies on the circle of radius $\frac{c}{2}$ with center C'. For the other points the calculation could be done in the same manner.

B4. From B2 it is easy to obtain: $A_{13}A_{12} = A_{13}C' + C'B' + B'A_{12} = \frac{c+a+b}{2} = p$; $A_{13}A_{03} = A_{13}A_{12} - A_{03}A_{12} = p - b$ (since $A_{03}A_{12}$ is a diameter of the circle from B2). Similarly, $A_{03}A_{02} = p - a$, $A_{02}A_{12} = p - c$.

B5. (One of the possible configurations — Problem 1.66 in the book of Prassolov, also see the article of V. Protassov ("Quant", № 4 — 2008); also see Problem 255 from the book of Sharygin 9 — 11, this Problem is specially mentioned in the Preface.)

From B1 and B2 it follows: $C'B_{23} \parallel AC$ и $C'B_{23} = C'A$, hence $\angle B_{23}AC' = \angle C'AB_{23} = \angle B_{23}AB_3$, thus AB_{23} is the external bisector of angle BAC. Moreover, from B2 it follows that $BB_{23} \perp AB_{23}$. Similarly for other points.

Comment. Note that Fig. contain many parallelograms (the sides of which are parallel either to the sides or to the bisectors of the triangles ABC). For example, taking parallelograms $A_3A_{13}A_{23}C$ and $BA_{13}A_{23}A_2$ we see another explanation of the equality from A1.

B6. Triangles $I_0A_{02}B_0$ and $I_0B_0C_{02}$ are similar (in the calculations of angles we use that B_0C_{02} is parallel to the external bisector of the angle ABC), hence $I_0A_{02} \cdot I_0C_{02} = r_0^2$.

B7#. The radical centers are points I_i and points symmetrical to them with respect to the circumcenter of triangle ABC.

For example, from B6 it follows that $I_0A_{02} \cdot I_0C_{02} = I_0A_{03} \cdot I_0B_{03} = I_0B_{01} \cdot I_0C_{01} = r_0^2$, hence I_0 has equal powers with respect to the circles with diameters A_0A_1, B_0B_2, C_0C_2 (see B1).

Then, consider, for instance, the circles with diameters A_2A_3, B_1B_3 and C_1C_2. The point I_3 lies on the radical axis of the first two circles, because the equal segments I_3A_2 and I_3B_1 are tangent lines to these circles. Moreover, the radical axis is perpendicular to the line joining the centers of the circles, i.e. the medial line of ABC. Three such lines intersect in the point symmetrical to I with respect to O.

B8#. These are the circles with centers I_i'.

Let X be the projection of I_0' to $B'C'$. Then from B3 it follows: $XA_{12} = XB' + B'A_{12} = \frac{p-b}{2} + \frac{b}{2} = \frac{p}{2}$. Further, $I_0'A_{12}^2 = I_0'X^2 + XA_{12}^2 = \frac{r_0^2 + p^2}{4}$. Similarly, the square of the distance from I_0' to each of the points A_{03}, A_{02}, C_{02}, C_{23}, B_{23}, B_{03} equals to $\frac{r^2 + p^2}{4}$.

In the same way it is proved that the radius of the circle with center I_1' equals to $\frac{r_1^2 + (p-a)^2}{4}$, etc.

Comment. The following general result holds: three pair of foci for three circles which centers are not collinear lie on a circle (the proof is an exercise on a power of a point with respect to a circle).

Comment. This circle is of so called *Tucker* circles for the triangle $I_1I_2I_3$.

B9#. (Also see the article of V. Protassov in "Quant" № 4 — 2008, this problem plays an important role in the proof of Feuerbach Theorem.) C_0A_{02} is a bisector of angle $AB'H_a$ (this follows from symmetry). Consider a nine-point circle, triangle $B'H_aH_b$ is inscribed to this circle, C' is a midpoint of the arc H_aH_b. Since (see B3) $C'A_{02} = C'H_a = C'H_b$ we have that A_{02} is a center of either inscribed or exscribed circle of triangle $B'H_aH_b$.

Series C: The Third Dozen: Intersections on the Altitudes

C1-3. From B5 it follows that $AA_{02}A_0A_{03}$ is a parallelogram (its sides are parallel to bisectors of angles CBA and ACB). Also $A_{(0)}A_{02}I_0A_{03}$ is a parallelogram (its sides are parallel to bisectors of angles CBA and ACB). Therefore $\overrightarrow{I_0A_0}$ and $\overrightarrow{A_{(0)}A}$ are symmetric with respect to the midpoint of the segment $A_{02}A_{03}$. This implies C1 and C2. Since $I_0A_0A_{(0)}A$ is a parallelogram, $A_{(i)}A_i \parallel AI_0$.

C4. (This is the Problem of Emelyanov No 10.7 from All-Russian Olympiad — 2002?.) From C3 it follows that these lines are the altitudes of the triangle $A_{(1)}B_{(2)}C_{(3)}$.

Note. One can show that the intersection point of these three lines is symmetric to the orthocenter of triangle $A_0B_0C_0$ with respect to I_0'.

C5. Show that the centers are points I_i'.

Radical axis bisects the segments of common tangents, hence from A3 it follows that B_2C_2 ($= B_2C_{(0)}$) and B_3C_3 ($= {}_3B_{(0)}$) are symmetruc with respect to the line $A'I_0'$, and also with respect to point I_0'. Similarly, $A_1C_{(0)}$ and ${}_3A_{(0)}$ are symmetric with respect to I_0'. This means that the corresponding points of intersection $C_{(0)}$ and C_3 are symmetric with respect to I_0'.

C6. The center is H.

From C1 we know that, for example, that $A_{(0)} = A_3C_3 \cap AH_a$ and $C_{(2)} = A_3C_3 \cap CH_c$. Since A_3C_3 is parallel to the bisector of the angle B, the angles between A_3C_3 and the altitudes AH_a and CH_c are equal. From that it follows that triangle $HA_{(0)}C_{(2)}$ is equilateral, hence H is equidistant from $A_{(0)}$ and $C_{(2)}$.

C7-8. The radii of the circumcircles from C6 equal $|\rho_i|$, where $\rho_0 = AH + r_1 = BH + r_2 = CH + r_3$, $\rho_1 = r_0 - AH = BH - r_3 = CH - r_2$, $\rho_2 = AH - r_3 = r_0 - BH = CH - r_1$, $\rho_3 = AH - r_2 = BH - r_1 = r_0 - CH$ (here AH, ect., could be negative if the corresponding angle of the triangle is obtuse). From this we have $AH = \dfrac{r_0 + r_1 + r_2 + r_3}{2} - r_1$, $BH = \dfrac{r_0 + r_1 + r_2 + r_3}{2} - r_2$, $CH = \dfrac{r_0 + r_1 + r_2 + r_3}{2} - r_3$. Further, $\rho_0 = \dfrac{r_0 + r_1 + r_2 + r_3}{2}$, $\rho_1 = \dfrac{r_0 + r_1 - r_2 - r_3}{2}$, $\rho_2 = \dfrac{r_0 - r_1 + r_2 - r_3}{2}$, $\rho_3 = \dfrac{r_0 - r_1 - r_2 + r_3}{2}$, and putting the relation $r_1 + r_2 + r_3 = 4R + r_0$ (see the book of Prassolov, Problem 12.24), $\rho_0 = r_0 + 2R$, $\rho_1 = |r_1 - 2R|$, $\rho_2 = |r_2 - 2R|$, $\rho_3 = |r_3 - 2R|$.

C9.# From A5 it follows that that I_0A' intersects the altitude AH_a at point S such that $\overrightarrow{AS} = \overrightarrow{I_0A_0}$, that is the point $A_{(0)}$ (see Problems C1-2).

C10.# (This Problems was proposed by D. Prokopenko) 1. It is easy to show that A is the midpoint of MN, hence $AA_{(0)}$ is the perpendicular bisector of MN.

2. From C3 it follows that $A_0A_{(0)}$ and A_0I_0 are symmetric with respect to the bisector of the angle MA_0N. Since A_0I_0 is the altitude of the triangle, $A_0A_{(0)}$ contains the circumcenter of triangle MA_0N.

Combining 1 and 2 we get the required statement.

The orthocenter of triangle A_0MN is the point symmetric to A_0 with respect to I_0.

The tasks of series D are the reformulation of the Emelyanovs' Theorem, and their solutions can be found in the book "Summer Conferences of the Tournament of Towns. Selected matherials. Volume 1." (MCCME, 2009, in Russian)